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A nonperturbative test of $M2$-brane theory

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Abstract: We discuss non-perturbative effects in the ABJM model due to monopole instantons. We begin by constructing the instanton solutions in the U(2) × U(2) model, explicitly, and computing the Euclidean action. The Wick-rotated Lagrangian is complex and its BPS monopole instantons are found to be a delicate version of the usual 't Hooft-Polyakov monopole solutions. They are generically 1/3 BPS but become 1/2 BPS at special locus in the moduli space of two M2-branes, yet each instanton carries eight fermionic zero modes, regardless of the vacuum choice. The low energy effective action induced by monopole instantons are quartic order in derivatives. The resulting vertices are nonperturbative in $1/k$, as expected, but are rational functions of the vacuum moduli. We also analyze the system of two M2-branes in the supergravity framework and compute the higher order interactions via 11-dimensional supergraviton exchange. The comparison of the two shows that the instanton vertices are precisely reproduced by this M2-brane picture, supporting the proposal that the ABJM model describes multiple M2-branes.

Keywords: Chern-Simons Theories, Nonperturbative Effects, M-Theory.

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1. Introduction and summary

Understanding the worldvolume dynamics of M2-branes is an important step in the study of M-theory. As a particularly interesting application, the superconformal field theory on the worldvolume of multiple M2-branes is believed to give a holographic description of the eleven-dimensional quantum supergravity on $AdS_4 \times S^7$. A supergravity analysis showed [1] that the number of degrees of freedom on N M2-branes scales like $N^{3/2}$, which implies a nontrivial interaction between the coincident M2-branes. This peculiar scaling property was believed to show up in the infrared strong coupling limit of the super Yang-Mills theory on N D2-branes, although so far we have been unable to get a precise understanding of its origin from the microscopic viewpoint.

It has been realized that the Chern-Simons gauge theories can have higher supersymmetries than the familiar $\mathcal{N} = 3$ barrier once the Yang-Mills term is turned off, and the resulting Chern-Simons-matter theories may have applications to multiple M2-branes. Especially, a maximally supersymmetric Chern-Simons matter theory has been constructed by Bagger, Lambert [2–4] and Gustavsson [5, 6] based on a mathematical structure called 3-algebra. On the other hand, another series of works [7–10] based on a more conventional approach have led to the full classification of Chern-Simons matter theories with $\mathcal{N} = 4, 5, 6$ supersymmetry. See also [11 − 14].

A particularly interesting example, called the ABJM model [10], is an $\mathcal{N} = 6$ superconformal Chern-Simons matter theory where the $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge fields of Chern-Simons level $(k, -k)$ are coupled to bi-fundamental matters. Aharony et.al. [10] proposed that this model is the worldvolume theory of N M2-branes in the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. There are a number of evidences supporting this proposal from the analysis of vacuum moduli space, brane construction, etc. Further analysis has been made on its mass deformation [15 − 17, 9] and the effect of fractional M2-branes [18]. Recently, a perfect agreement of the superconformal index between the field theory and the dual supergravity was found in a certain limit [19], and further evidences supporting the proposal have been found in the integrability structure of the two theories [20 − 29].

In this paper we make a first step to understand the quantum correction in the ABJM model at the nonperturbative level, as a rather nontrivial test of the proposal. In particular we consider instanton processes in the field theory side and identify their counterpart in the dual 11-dimensional supergravity approach. To summarize the result first, we find that instantons in the ABJM theories are of monopole type with eight fermionic zero modes each, and that the instanton processes generate a series of higher order interaction terms in the Coulomb phase. These range from a four-derivative bosonic terms to eight fermion vertices. We also find that these higher order correction terms have a well-understood origin in terms of M2-branes interacting via supergravity and thereby compute the bulk counterpart accurately. Finally we show that the scaling behavior of the latter matches precisely the effective and nonperturbative Lagrangian we computed from the monopole instanton, which suggests strongly that this ABJM theory is indeed the worlvolume theory of multiple M2-branes.

There hasve been similar considerations for three-dimensional $\mathcal{N} = 8$ Yang-Mills theory [30]. Here the monopole instanton corrections were interpreted in the supergravity side as exchanges of D0-branes between a pair of M2-branes transverse to the M-theory circle, or equivalently between a pair of D2-branes. Structures of the resulting higher order corrections were determined quite precisely [31 − 33], and the match between the Yang-Mills side and the M-theory side were demonstrated convincingly.

There are some notable differences between these two cases. From the gravity side, the main difference is in the eleven-dimensional backgrounds. The former has two M2-branes in $\mathbb{R}^8/\mathbb{Z}_k \times \mathbb{R}^{2+1}$, while the latter has two M2-branes (transverse to S^1) in $S^1 \times \mathbb{R}^7 \times \mathbb{R}^{2+1}$. The D0-branes, which are the bulk counterpart of the Yang-Mills monopole instantons in the latter, must be now reinterpreted in the orbifold case, given the absence of a topological circle, as one of the angular momentum in $\mathbb{R}^8$. The angular momentum in question turned out to be along the direction of the orbifolding action $\mathbb{Z}_k$.

In the field theory side, the difference runs much deeper. Since the ABJM theory contains a pair of Chern-Simons terms, one generally expects a rather different behavior of monopole instantons, if there is any. For instance, the Wick-rotated Lagrangian for such theories is not real since the Chern-Simons term acquire a factor of i. In part due to this, one generically finds that some real fields take complex configurations for the saddle point.

However, this is not really a problem as long as the solution is regular and converges to the (real) vacuum asymptotically. Using a complex saddle point here is no different than using a complex saddle point when we perform ordinary contour integral of a function with critical points off the real axis. As long as we make sure the semi-classical configuration approaches the correct (real) vacuum and as long as we take care not to over-count excitations around this Euclidean solution, this is a right thing to do.

Another, potentially more serious, worry arises from the gauge variance of the classical action. In monopole backgrounds of any Chern-Simons theories, asymptotically nontrivial gauge transformations shift the Euclidean action by some imaginary constants. As was argued in [34], naive integration over this gauge orbit seems to project out the amplitudes involving nonzero number of monopole-instantons. We show, however, that this argument is misleading. The gauge variance of the action simply means that the monopole-instanton carries the unbroken gauge charge, and that it mediates transitions between states with different charges [35]. Gauge variance of monopole action cancels against the gauge variance due to the two mismatching electric charges in the initial and the final wave functions, so that the transition amplitudes are gauge invariant as a whole.

Those who are familiar with Chern-Simons theories may wonder whether there is a finite action monopole instanton at all, since, for example, generic Chern-Simons Yang-Mills theories are massive gauge theory and cannot have finite action monopole instantons. There is a well-known linear divergence. If such a behavior were found here, this by itself would have ruled out the ABJM model as a theory of M2-branes. Fortunately, however, there is no such divergence here. In fact, the monopole instanton solutions here are essentially the usual BPS monopoles of Yang-Mills theory up to a complexified gauge rotation. See section 3 and 5 for the explicit forms. Our 'complexified' monopole instantons are novel and original. Their nonperturbative effect remains to be explored in less supersymmetric varieties of the ABJM type theories.

Our M-theory dual calculation leads to a very detailed and precise effective Lagrangian for the M2-branes and contains both perturbative and nonperturbative corrections when viewed from the field theory side. We have reproduced the correct scaling behaviors of those corresponding to the nonperturbative parts by studying monopole instantons, but stopped short of computing loop corrections to these interaction vertices, such as loop correction to the monopole instanton saddle point. Nor did we try to evaluate the simple perturbative loop corrections, which according to the M-theory computation, should also begin at the four-derivative level. It would be interesting to reproduce the entire structure and the coefficients of M-theory result, from a purely field theoretic calculation of the ABJM model.

In section 2, we start with a brief review of the ABJM model, focusing especially on its vacuum moduli space. We present generic vacua of the theory with gauge group $\mathrm{U}(2)\times\mathrm{U}(2)$ and their massive spectrum. Then we turn to study the monopole equations in $\mathrm{U}(2)\times\mathrm{U}(2)$ model. We are able to find the 'BPS monopole instanton configuration' throughout the vacuum moduli space by simple embedding of the well known 't Hooft Polyakov solution. Our stationary monopole solution is generically complex. This monopole solution is 1/3 BPS at generic points in moduli space, while at some special locus it becomes 1/2 BPS and

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takes a simpler form. In section 3 we first give the construction of the 1/2 BPS solutions in the special cases where only one complex scalar field takes nonzero vacuum expectation value, and then present the general solutions in section 5.

In section 4 we get a simple expression for the Euclidean action, and find that each monopole-instanton carries eight fermion zero-modes. Although the action is not invariant under a certain gauge transformation, it simply reflects the gauge charge carried by the monopole and does not mean their effects are projected out. In section 5, we also calculate the monopole action and the zero-modes for the general 1/3 BPS multi-monopole instantons.

The issues on gauge invariance in Chern-Simons theories will be explained in greater detail in section 6. Later in that section we also argue that the instanton effects are described by local vertex operators in the low-energy moduli dynamics, and discuss several constraints on their possible forms.

Finally, in section 7 we move to the M/string theory framework. We first study the system of two M2-branes in supergravity and see the correspondence between certain transverse momentum exchanges between the M2-branes and the multi-monopole instanton processes in the ABJM model. We then turn to the type IIA picture and show that the D0-brane exchange along the Euclidean geodesic line between two D2-branes reproduce the monopole instanton action of the field theory. We also get the correct mass spectrum in the generic vacua of the field theory from the energy of the fundamental string connecting two D2-branes. These agreements between the ABJM model and the dual supergravity provide strong evidences that the ABJM model is the correct theory of M2-branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$.

In appendix A we take the simple example of abelian BF-matter theory, and give the explicit construction of the so-called 0-cocycle which is necessary to make the action gauge invariant. This is complementary to the abstract discussion of cocycles given in section 6. In appendix B, we recalculate the Euclidean action of monopole instantons by somewhat different approach from section 4. In appendix C, we recapitulate the monopole vertex operators in the Maxwell theory and in the Chern-Simons matter theory for further clarification.

2. The ABJM model

We present in this section a short description on the ABJM model [10], believed to describe the dynamics of multiple M2-branes probing a certain orbifold geometry. This $\mathcal{N} = 6$ supersymmetric model has the gauge symmetry $G = \mathrm{U}(N)_1 \times \mathrm{U}(N)_2$ whose gauge fields are denoted by A_μ and $\tilde{A}_\mu$ with the Chern-Simons kinetic term of level $(k, -k)$. The matter fields are composed of four complex scalars Z_α ($\alpha = 1, 2, 3, 4$) and four three-dimensional spinors Ψ^α, both of which transform under G as $(\mathbf{N}, \bar{\mathbf{N}})$. As well as the gauge symmetry, the present model also has additional global SU(4) R-symmetry, under which the scalars Z_α furnish the representation $\mathbf{4}$ while the fermions Ψ^α furnish $\bar{\mathbf{4}}$.

Let us start with the Lagrangian of the ABJM model,

$$\mathcal{L} = \mathcal{L}_{\mathrm{CS}} + \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{potential}}\,, \tag{2.1}$$

where

$$
\begin{aligned}
\mathcal{L}_{\rm CS}+\mathcal{L}_{\rm kin} &= \frac{k}{4\pi}\epsilon^{\mu\nu\rho}{\rm tr}\big(A_\mu\partial_\nu A_\rho - i\frac{2}{3}A_\mu A_\nu A_\rho - \tilde{A}_\mu\partial_\nu\tilde{A}_\rho + i\frac{2}{3}\tilde{A}_\mu\tilde{A}_\nu\tilde{A}_\rho\big) \\
&\quad -{\rm tr}\left(D_\mu\bar{Z}^\alpha D^\mu Z_\alpha - i\bar{\Psi}_\alpha\gamma^\mu D_\mu\Psi^\alpha\right)\,, \\
\mathcal{L}_{\rm Yukawa} &= -\frac{2\pi i}{k}{\rm tr}\left(\bar{Z}^\alpha Z_\alpha\bar{\Psi}_\beta\Psi^\beta - Z_\alpha\bar{Z}^\alpha\Psi^\beta\bar{\Psi}_\beta + 2\bar{Z}^\alpha\Psi^\beta\bar{\Psi}_\alpha Z_\beta - 2Z_\alpha\bar{\Psi}_\beta\Psi^\alpha\bar{Z}^\beta\right) \\
&\quad -\frac{2\pi i}{k}\epsilon^{\alpha\beta\gamma\delta}{\rm tr}\left(Z_\alpha\bar{\Psi}_\beta Z_\gamma\bar{\Psi}_\delta\right) + \frac{2\pi i}{k}\epsilon_{\alpha\beta\gamma\delta}{\rm tr}\left(\bar{Z}^\alpha\Psi^\beta\bar{Z}^\gamma\Psi^d\right)\,, \qquad (2.2)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{L}_{\rm potential} = +\frac{4\pi^2}{3k^2}{\rm tr}\Big(&Z_\alpha\bar{Z}^\alpha Z_\beta\bar{Z}^\beta Z_\gamma\bar{Z}^\gamma + \bar{Z}^\alpha Z_\alpha\bar{Z}^\beta Z_\beta\bar{Z}^\gamma Z_\gamma \\
&+4Z_\alpha\bar{Z}^\gamma Z_\beta\bar{Z}^\alpha Z_\gamma\bar{Z}^\beta - 6Z_\alpha\bar{Z}^\alpha Z_\beta\bar{Z}^\gamma Z_\gamma\bar{Z}^\beta\Big)\,. \qquad (2.3)
\end{aligned}
$$

We basically use the convention of [9] except the hermitian gauge fields so that the covariant derivatives now become

$$
D_\mu Z_\alpha = \partial_\mu Z_\alpha - iA_\mu Z_\alpha + iZ_\alpha\tilde{A}_\mu\,, \qquad (2.4)
$$

and Chern-Simons level k is now quantized as an integer, i.e., $k\in\mathbb{Z}$. The trace is over $N\times N$ matrices of either gauge group and leaves the gauge invariant quantities. The contraction of spinor fields is the standard one. This Lagrangian is invariant under the $\mathcal{N}=6$ supersymmetry whose transformation rules are

$$
\begin{aligned}
\delta Z_\alpha &= -i\eta_{\alpha\beta}\Psi^\beta, \qquad (2.5)\\
\delta\Psi^\alpha &= \left[\gamma^\mu D_\mu Z_\gamma - \frac{4\pi}{3k}(Z_{[\beta}\bar{Z}^\beta Z_{\gamma]})\right]\eta^{\gamma\alpha} + \frac{8\pi}{3k}(Z_\beta\bar{Z}^\alpha Z_\gamma)\eta^{\gamma\beta} - \frac{4\pi}{3k}\epsilon^{\alpha\beta\gamma\delta}(Z_\beta\bar{Z}^\rho Z_\gamma)\eta_{\delta\rho}\,,\\
\delta A_\mu &= \frac{2\pi i}{k}(\eta^{\alpha\beta}\gamma_\mu Z_\alpha\bar{\Psi}_\beta + \eta_{\alpha\beta}\gamma_\mu\Psi^\beta\bar{Z}^\alpha),\quad \delta\tilde{A}_\mu = \frac{2\pi i}{k}(\eta^{\alpha\beta}\gamma_\mu\bar{\Psi}_\beta Z_\alpha + \eta_{\alpha\beta}\gamma_\mu\bar{Z}^\alpha\Psi^\beta)\,,
\end{aligned}
$$

where the transformation parameters $\eta^{\alpha\beta}$ satisfy the relations

$$
\eta^{\alpha\beta} = -\eta^{\beta\alpha}, \qquad \eta_{\alpha\beta} = (\eta^{\alpha\beta})^* = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\eta^{\gamma\delta}\,. \qquad (2.6)
$$

Let us now examine the vacuum moduli space of the present model at the classical level, i.e., solutions of $V(\Phi)=0$ up to gauge transformations. It is known that the potential can be made into a sum of squares

$$
V = \frac{2\pi^2}{3k^2}{\rm tr}\left(W_{\alpha\gamma}^{\ \ \beta}\bar{W}^{\gamma\alpha}_{\ \ \beta}\right) \qquad (2.7)
$$

with

$$
\begin{aligned}
W_{\alpha\gamma}^{\ \ \beta} &= (2Z_\alpha\bar{Z}^\beta Z_\gamma - \delta^\beta_\gamma Z_\alpha\bar{Z}^\rho Z_\rho - \delta^\beta_\alpha Z_\rho\bar{Z}^\rho Z_\gamma) - (\alpha\leftrightarrow\gamma)\,,\\
\bar{W}^{\alpha\gamma}_{\ \ \beta} &= (2\bar{Z}^\alpha Z_\beta\bar{Z}^\gamma - \delta^\gamma_\beta\bar{Z}^\alpha Z_\rho\bar{Z}^\rho - \delta^\alpha_\beta\bar{Z}^\rho Z_\rho\bar{Z}^\gamma) - (\alpha\leftrightarrow\gamma)\,, \qquad (2.8)
\end{aligned}
$$

which leads to the equation for its minima

$$
Z_\alpha\bar{Z}^\beta Z_\gamma = Z_\gamma\bar{Z}^\beta Z_\alpha\,, \quad \bar{Z}^\alpha Z_\beta\bar{Z}^\gamma = \bar{Z}^\gamma Z_\beta\bar{Z}^\alpha\,. \qquad (2.9)
$$

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This implies that the hermitian matrices $Z_\alpha \bar{Z}^\beta$ commute with each other, and similarly for $\bar{Z}^\alpha Z_\beta$. The vacuum solutions are thus given by diagonal Z_α up to gauge equivalences,

$$Z_\alpha = \mathrm{diag}(z_\alpha^1, z_\alpha^2, \ldots, z_\alpha^N) \; . \tag{2.10}$$

On a generic point of the vacuum moduli space, the gauge group $G = \mathrm{U}(N) \times \mathrm{U}(N)$ is spontaneously broken down to $\mathrm{U}(1)^N \subset \mathrm{U}(N)_D$, diagonal part of G.

In order to describe a classical Lagrangian that governs the dynamics of massless moduli fields, we first take the diagonal elements of gauge fields A_μ and $\tilde{A}_\mu$, i.e.,

$$A_\mu = \mathrm{diag}(a_\mu^1, a_\mu^2, \ldots, a_\mu^N), \qquad \tilde{A}_\mu = \mathrm{diag}(\tilde{a}_\mu^1, \tilde{a}_\mu^2, \ldots, \tilde{a}_\mu^N) \; . \tag{2.11}$$

Although $a^i - \tilde{a}^i$ are the gauge fields of the broken gauge symmetries, we need to keep them [36, 37]. In terms of these diagonal variables, the classical low-energy Lagrangian is

$$\mathcal{L}_{\mathrm{cl}} = -\sum_i |D_\mu z_\alpha^i|^2 + \sum_i \frac{k}{4\pi} \epsilon^{\mu\nu\rho} (a_\mu^i - \tilde{a}_\mu^i) f_{\nu\rho}^i \, , \tag{2.12}$$

where $D_\mu z_\alpha^i = \partial_\mu z_\alpha^i - i(a_\mu^i - \tilde{a}_\mu^i) z_\alpha^i$ and $f^i = d(a^i + \tilde{a}^i)/2$. The role of the Chern-Simons terms for the moduli dynamics can be seen best by dualizing $(a_\mu^i + \tilde{a}_\mu^i)/2$. This is done by adding to $\mathcal{L}_{\mathrm{cl}}$ a term

$$\mathcal{L}_{\mathrm{dual}} = -\frac{1}{4\pi} \epsilon^{\mu\nu\rho} \sum_i \partial_\mu \theta^i f_{\nu\rho}^i \, , \tag{2.13}$$

and by treating f^i as the fundamental variable. The θ^i variables are normalized to have period 2π. Integrating over θ^i brings us back to the original low energy Lagrangian, whereas integrating over f^i imposes the condition,

$$k(a_\mu^i - \tilde{a}_\mu^i) = \partial_\mu \theta^i \; . \tag{2.14}$$

The Chern-Simons terms disappear upon this, while the kinetic term simplifies to an ordinary linear sigma model

$$\mathcal{L} = -\left|\partial_\mu \tilde{z}_\alpha^i\right|^2, \qquad \tilde{z}_\alpha^i = e^{-i\theta^i/k} z_\alpha^i \; . \tag{2.15}$$

Note that $\tilde{z}_\alpha^i$ are invariant under local gauge transformations. The 2π periodicity of θ^i, combined with the Weyl symmetry, tells us that the vacuum moduli space is an orbifold

$$\left(\mathbb{C}^4/\mathbb{Z}_k\right)^N / S_N \, , \tag{2.16}$$

and also that the correct low energy variables to use are these invariant fields $\tilde{z}_\alpha^i$ [10]. These gauge invariant moduli coordinates $\tilde{z}_\alpha^i, i = 1, \ldots, N$ denote the positions of N M2-branes on the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ after a proper scaling.

Let us now in turn discuss the vacuum degeneracy of the theory. Since we can add to a given ground state, without costing any energy, the magnetic flux f_{12}^i together with certain amounts of charges, the theory has huge number of vacuum degeneracy. Here the flux and charges that specify the vacuum should satisfy the Gauss laws of the model (2.12),

$$\begin{aligned} \frac{k}{2\pi} f^i_{12} - i(D_0 z^i_\alpha \bar{z}^{i\alpha} - z^i_\alpha D_0 \bar{z}^{i\alpha}) &= 0\,, \\ \partial_1(a^i_2 - \tilde{a}^i_2) - \partial_2(a^i_1 - \tilde{a}^i_1) &= 0\,. \end{aligned} \tag{2.17}$$

Magnetic monopole instantons are those which interpolate between vacua of different magnetic flux and charges. The monopole instantons thus violate some of the global charges in the vacuum moduli dynamics (2.15). In section 6 we will construct the local vertex operators describing their effect using the gauge invariant variables $\tilde{z}^i_\alpha$.

We close this section with mass spectrum on the generic point of vacuum moduli space. For an instance, let us consider the vacua of the theory with $\mathrm{U}(2)\times\mathrm{U}(2)$ gauge group. By the $\mathrm{SU}(4)_{\mathrm{R}}$ and gauge transformations, one can parameterize them as

$$\langle Z_1\rangle = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \qquad \langle Z_2\rangle = \begin{pmatrix} cu_2 & 0 \\ 0 & cu_1 \end{pmatrix}, \qquad \langle Z_3\rangle = \langle Z_4\rangle = 0\,, \tag{2.18}$$

where the parameters are all real and obey $0 < u_1 < u_2$ and $0 < c$. Note that the two M2-brane are at $z^1_\alpha = (u_1, cu_2, 0, 0)$ and $z^2_\alpha = (u_2, cu_1, 0, 0)$. The linear fluctuation analysis tells us that the mass spectrum in this vacuum is given by

$$\begin{aligned} \text{massless multiplet} &: 16 \text{ scalar bosons} + 16 \text{ fermions}, \\ \text{massive multiplet} &: 12 \text{ scalar bosons} + 16 \text{ fermions} + 4 \text{ vector bosons}, \end{aligned} \tag{2.19}$$

where the mass of the massive multiplet is

$$\begin{aligned} \mu &= \frac{2\pi}{|k|}\sqrt{\left((z^1\cdot\bar{z}^1)^2 + (z^2\cdot\bar{z}^2)^2\right)^2 - 4\left|z^1\cdot\bar{z}^2\right|^2} \\ &= \frac{2\pi}{|k|}(1+c^2)(u_2^2 - u_1^2)\,. \end{aligned} \tag{2.20}$$

This agrees with the result in [38]. Here dot indices denote the $\mathrm{SU}(4)_{\mathrm{R}}$ indices contraction. The spin structure of the massive multiplet is $(1, \frac{1}{2}, 0, -\frac{1}{2}, -1)$ with multiplicity $2\times(1,4,6,4,1)$. In section 7, we interpret the vacuum expectation value (2.18) as the positions of two M2-branes in $\mathbb{C}^4/\mathbb{Z}_k$, and the M2-brane connecting these two branes has the energy given by the above mass formula.

3. Monopole instantons and the reality condition

In this section, we wish to look for monopole instanton solutions. For the instanton physics, we consider the Euclidean version of the theory. As usual, we take the Wick rotation $t = -i\tau$ to obtain the Euclidean Lagrangian,

$$-\mathcal{L}_{\mathrm{E}} = i\mathcal{L}_{\mathrm{CS}} + \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{Yukawa}} + \mathcal{L}_{\mathrm{potential}}\,. \tag{3.1}$$

It is noteworthy here that the Chern-Simons coupling $\mathcal{L}_{\rm CS}$ gets the imaginary sign which will introduce several subtle issues in later sections. Three-dimensional Euclidean gamma matrices γ^μ are chosen to satisfy the relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \qquad \gamma^{\mu\nu\rho} = i\epsilon^{\mu\nu\rho} \ . \tag{3.2}$$

In the most of this work we focus on the case with $\mathrm{U}(2) \times \mathrm{U}(2)$ gauge group, which is the simplest where monopole instantons appear. We will work with the parametrization of the vacua given in (2.18). Let us begin with the special case $c = 0$ where only one of the four scalars takes non-zero vev, say $Z = Z_1$,

$$\langle Z \rangle = U \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} V^{-1} = \langle \bar{Z} \rangle^\dagger \tag{3.3}$$

for some unitary U and V. $\langle \bar{Z} \rangle$ is of course the conjugate of $\langle Z \rangle$, so the latter equation is redundant. The reason we show it explicitly should become clear in a moment. Without loss of generality, we suppose that $u_{1,2}$ are real and that $0 < u_1 < u_2$. In terms of the M2-brane interpretation, these two are radial positions of the two M2-branes in the orbifold $\mathbb{C}^4/\mathbb{Z}_k$.

We are looking for a monopole instanton that preserves some supersymmetry. The BPS equation coming from supersymmetry transformation is pretty simple when we turn on only one of the four scalar fields, and with

$$DZ \equiv dZ - iAZ + iZ\tilde{A} \quad \text{and} \quad D\bar{Z} \equiv d\bar{Z} - i\tilde{A}\bar{Z} + i\bar{Z}A \,, \tag{3.4}$$

we have

$$DZ = 0 \quad \mathit{or} \quad D\bar{Z} = 0 \tag{3.5}$$

as the condition for half-BPS configurations. One would think that the second equation is the same as the first, again since $\bar{Z}$ is merely a conjugate of Z, in which case this will certainly lead to constant Z and $\bar{Z}$ only.

However, the ABJM model is a Chern-Simons theory. The Chern-Simons term acquires a factor i upon Wick rotation, and the Euclidean action becomes complex. In such circumstances, the saddle point evaluation can often involve deformation of the path-integral into complex planes of (what used to be real) field variables. In appendix B, such complexified stationary path is found for a very simple mechanical model. The semi-classical configurations that dominate the path integral need not satisfy the usual reality constraints. This is nothing new, and we do such deformation of contour all the time when we perform ordinary integration of real functions.

It may happen that there exists a saddle point where only one of the two conditions (3.5) is satisfied, say

$$DZ = 0 \ . \tag{3.6}$$

This is the type of saddle points we are interested in, and the solution we obtain can be interpreted as a monopole-like instanton.[1] The broken supersymmetry generators are

[1] We can treat $D\bar{Z}$ and DZ differently, in part because each enters the supersymmetry transformation rule of Ψ and $\bar{\Psi}$. In Euclidean signature, as is well known, these two fermions must be treated as independent variables, so their supersymmetry transformation can be treated independently as well.

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along $\eta_{12}, \eta_{13}, \eta_{14}$ for this case.[2] As it will become clear soon, the other choice $D\bar{Z} = 0$ with broken supersymmetry generators along $\eta^{1\alpha}$ corresponds to anti-monopole solution.

Using $DZ = 0$ together with the Gauss constraints for A and $\tilde{A}$, we find the following set of equations

$$\frac{k}{2\pi} * F \equiv \frac{k}{2\pi} * (dA - iA \wedge A) = -D(Z\bar{Z}) ,$$
$$\frac{k}{2\pi} * \tilde{F} \equiv \frac{k}{2\pi} * (d\tilde{A} - i\tilde{A} \wedge \tilde{A}) = -D(\bar{Z}Z) . \tag{3.7}$$

Note that

$$D{*}D(Z\bar{Z}) = 0 = D{*}D(\bar{Z}Z) \tag{3.8}$$

follows by a further use of the Bianchi identity, so the BPS equation together with the Gauss constraint implies the equation of motion

$$D{*}D\bar{Z} = 0 \tag{3.9}$$

as long as the covariantly constant Z is nonsingular.

The master equations (3.7) look like ordinary BPS equation for monopoles. As an initial attempt, let us consider $A = \tilde{A}$, so that $F = \tilde{F}$. The BPS equation then implies $D \wedge DZ = -i[F, Z] = 0$, which together with (3.7) forces (with some constants a, b, c)

$$Z = c\mathbf{1}_2, \qquad \bar{Z} = a\,\Phi + b\mathbf{1}_2, \tag{3.10}$$

where Φ is a 2×2 traceless scalar function that, together with $A = \tilde{A}$, solves the ordinary monopole BPS equation. However, this has the asymptotic value $\langle Z\rangle^\dagger \neq \langle \bar{Z}\rangle$ which violates the reality condition, and, as such, is unusual. The only exception occurs when $a = 0, b^* = c$ which brings us back to a vacuum.

Underlying this difficulty is that the gauge fields $A = \tilde{A}$ in this ansatz is perfectly real, even though we do not expect the saddle point that obeys usual reality conditions. What we cannot abandon is the reality condition of the vacuum itself, so we must be prepared to trade off the (partial) reality of the instanton solution in favor of the reality of the scalar vev.

Motivated by this initial failure, let us consider the following redefinition of variables

$$Z = L\mathcal{Z}L, \quad \bar{Z} = L^{-1}\bar{\mathcal{Z}}L^{-1} \tag{3.11}$$

accompanied by cancelling transformation of the gauge fields,

$$A = L\mathcal{A}L^{-1} + iLdL^{-1} , \quad \tilde{A} = L^{-1}\tilde{\mathcal{A}}L + iL^{-1}dL , \tag{3.12}$$

none of which preserve the reality conditions. On the other hand, the BPS equation and the Gauss constraint are preserved, so $\mathcal{A}, \tilde{\mathcal{A}}, \mathcal{Z}, \bar{\mathcal{Z}}$ obey the same set of equations as $A, \tilde{A}, Z, \bar{Z}$. One can think of L as a complexified gauge transformation, although we are not suggesting it as a symmetry of the theory itself.

[2] Here we assume that the Euclidean supersymmetry parameters satisfy the reality condition similar to (2.6), implying twelve real supersymmetries in the Euclidean theory as well.

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The point of doing this redefinition is that now we can use the ansatz $\mathcal{A} = \tilde{\mathcal{A}}$ without worrying about the reality condition between $\langle \mathcal{Z} \rangle$ and $\langle \bar{\mathcal{Z}} \rangle$. The general solution with the reality condition $\langle Z \rangle = \langle \bar{Z} \rangle^\dagger$ asymptotically satisfied turns out to be

$$\mathcal{Z} = \sqrt{u_1 u_2}\mathbf{1}_2\,, \qquad \bar{\mathcal{Z}} = \frac{1}{\sqrt{u_1 u_2}}\left((u_1^2 - u_2^2)\,\Phi + \frac{u_1^2 + u_2^2}{2}\mathbf{1}_2 \right), \tag{3.13}$$

and

$$*\,\mathcal{F} \equiv *(d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}) \,=\, \mu\,(d\Phi - i[\mathcal{A}, \Phi])\;, \tag{3.14}$$

where Φ is normalized so that $\mathrm{tr}\langle\Phi\rangle^2 = 1/2$ and μ is the mass parameter (2.20) with $c = 0$,

$$\mu = \frac{2\pi}{k}\left(u_2^2 - u_1^2\right) > 0. \tag{3.15}$$

The equation (3.14) is nothing but the usual BPS monopole equation with the scale μ [39]. The solution for a single monopole is

$$\Phi = \left(\coth \mu r - \frac{1}{\mu r}\right)\frac{\hat{r}^a \sigma^a}{2}, \qquad \tilde{\mathcal{A}} = \mathcal{A} = \frac{1}{2}\left(\frac{\mu r}{\sinh \mu r} - 1\right)\epsilon^{abc}\sigma^a \hat{r}^b d\hat{r}^c\;. \tag{3.16}$$

One can reconstruct $A, \tilde{A}, Z, \tilde{Z}$ by finding appropriate transformation matrix L.

To find L, and also to see how (3.13) leads to the solution with physically acceptable vev, consider

$$Z = \sqrt{u_1 u_2}L^2, \quad \bar{Z} = L^{-1}\frac{1}{\sqrt{u_1 u_2}}\left((u_1^2 - u_2^2)\,\Phi + \frac{u_1^2 + u_2^2}{2}\mathbf{1}_2 \right)L^{-1}\;. \tag{3.17}$$

With real $u_{1,2}$ it is clear that $\langle Z \rangle = \langle \bar{Z} \rangle^\dagger$ can be satisfied for L of the general form

$$L = e^{\Lambda(x)\langle\Phi\rangle}\;, \tag{3.18}$$

where asymptotic value Λ_* of $\Lambda(x)$ is constant on S^2_∞. This value should be

$$e^{\Lambda_*} = \sqrt{\frac{u_1}{u_2}}. \tag{3.19}$$

To see this, we need to compare the asymptotic value at each point on S^2_∞. This can be easily done in the unitary gauge $\langle\Phi\rangle = \sigma_3/2$ where we have

$$\lim_{x\to\infty} L^2 = e^{\Lambda_* \sigma_3}\;, \tag{3.20}$$

and (3.17) leads to the vev

$$\langle Z \rangle = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} = \langle \bar{Z} \rangle^\dagger \tag{3.21}$$

as promised, up to gauge rotations U and V. One choice of L which is smooth everywhere is $L = e^{-\frac{1}{2}\log(u_2/u_1)\,\Phi(x)}$.

Finally, let us consider the other choice of BPS equation $D\bar{Z} = 0$. This choice leads to a different set of equations when combined with the Gauss constraints

$$\begin{aligned} \frac{k}{2\pi} * F &\equiv \frac{k}{2\pi} * (dA - iA \wedge A) \;=\; D(Z\bar{Z})\,, \\ \frac{k}{2\pi} * \tilde{F} &\equiv \frac{k}{2\pi} * (d\tilde{A} - i\tilde{A} \wedge \tilde{A}) \;=\; D(\bar{Z}Z)\;. \end{aligned} \tag{3.22}$$

The analog of (3.17) for the scalar field is now

$$Z = \tilde{L}\frac{1}{\sqrt{u_1 u_2}}\left((u_1^2 - u_2^2)\,\Phi + \frac{u_1^2 + u_2^2}{2}\mathbf{1}_2\right)\tilde{L}, \qquad \bar{Z} = \sqrt{u_1 u_2}\;\tilde{L}^{-2}, \tag{3.23}$$

with $A = \tilde{L}\mathcal{A}\tilde{L}^{-1} + i\tilde{L}d\tilde{L}^{-1}$, $\tilde{A} = \tilde{L}^{-1}\tilde{\mathcal{A}}\tilde{L} + i\tilde{L}^{-1}d\tilde{L}$, which leads us to the anti-BPS equation for ordinary monopoles

$$* \mathcal{F} \equiv *(d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}) \;=\; -\mu\,(d\Phi - i[\mathcal{A}, \Phi]) \tag{3.24}$$

with the same scale $\mu > 0$ as before. If we choose to write the anti-monopole instanton to have the same $\langle\Phi\rangle$ as that of the monopole instanton, $\tilde{L} = L^{-1}$ will do the trick for reconstruction of the anti-monopole instanton $A, \tilde{A}, Z, \bar{Z}$ from this data. What is important for us is that the two cases differ by $Z \leftrightarrow \bar{Z}$ and the relative sign change between $d\mathcal{A} - i\mathcal{A}^2$ and $(d - i\mathcal{A})\Phi$.

4. Euclidean action and zero-modes

There is a potential subtlety with the Euclidean action, because a Chern-Simons monopole mediates two states that differ by k units of electric charge. When the transition is not vacuum-to-vacuum one, the computation of the WKB amplitude can in general involve the so-called cocycle factor. However, at the end of the day the ordinary Euclidean action would suffice with the present solution, as far as the modulus of the WKB amplitude goes, so let us evaluate S_{E} for our solutions. The cocycle issues will be addressed in section 6 and appendix A. An alternative evaluation of the monopole action is given in appendix B.

S_{E} has three bosonic pieces, the Chern-Simons term, the scalar kinetic term, and the potential term. The potential term does not contribute since only one complex scalar is turned on, while the scalar kinetic term, $D\bar{Z}^\alpha DZ_\alpha$, vanishes on either of BPS or anti-BPS equations, $DZ = 0$ or $D\bar{Z} = 0$. Thus, the only piece that contributes is the Euclidean Chern-Simons action. For a monopole instanton, therefore, we find

$$\begin{aligned} -S_{\mathrm{E}} &= \frac{ik}{4\pi}\int \Big(\omega_3(A) - \omega_3(\tilde{A})\Big) \\ &= \frac{ik}{4\pi}\int \Big(\omega_3(L\mathcal{A}L^{-1} + iLdL^{-1}) - \omega_3(L^{-1}\tilde{\mathcal{A}}L + iL^{-1}dL)\Big)\,. \end{aligned} \tag{4.1}$$

This can be split into pieces involving $\mathcal{A} = \tilde{\mathcal{A}}$ only, which cancel each other, and the rest

$$-S_{\mathrm{E}} = \frac{k}{4\pi}\int_{S^2_\infty}\left(\int_0^1 ds\,\mathrm{tr}[\log(L)dA_s]\right) + \frac{k}{4\pi}\int_{S^2_\infty}\left(\int_0^1 ds\,\mathrm{tr}[\log(L)d\tilde{A}_s]\right) \tag{4.2}$$

with $A_s \equiv L^s \mathcal{A} L^{-s} + i L^s dL^{-s}$ and $\tilde{A}_s \equiv L^{-s} \mathcal{A} L^s + i L^{-s} dL^s$. Thus, it suffices to understand the asymptotic behavior of the gauge fields.

Parameterizing $\langle\Phi\rangle$ as $n^a\sigma^a/2$ with a unit 3-vector n, the asymptotic gauge field has the form,

$$\mathcal{A}\Big|_{S^2_\infty} = \frac{\sigma^a}{2}(dn \times n + \alpha n)^a \,, \tag{4.3}$$

where the cross product is with respect to the SU(2) adjoint indices and α is a 1-form. This comes from $D\Phi = O(1/r^2)$. The asymptotic forms of dA_s and $d\tilde{A}_s$ are such that

$$n^a d\mathcal{A}^a\Big|_{S^2_\infty} = n^a dA^a_s\Big|_{S^2_\infty} = n^a d\tilde{A}^a_s\Big|_{S^2_\infty} = n^a(-dn \times dn)^a + d\alpha \tag{4.4}$$

regardless of L^s, since the transformation by L only shifts α by $\pm i d\Lambda$. It is instructive to consider first the asymptotic form of $n^a\mathcal{F}^a$,

$$n^a\mathcal{F}^a\Big|_{S^2_\infty} = \frac{1}{2} n^a(-dn \times dn)^a + d\alpha \,. \tag{4.5}$$

Note the relative factor 1/2 in front of the two first terms in the two expressions. Recall that the monopole solution is such that

$$\int_{S^2_\infty} n^a\mathcal{F}^a = 4\pi \tag{4.6}$$

by definition. For the spherically symmetric Hedge-Hog gauge with $n^a = -\hat{r}^a$ and $\alpha = 0$, this can be seen explicitly by integrating the first term of (4.5). For more general but still smooth gauge choice, the first term yields the same 4π since it is a topological expression while $d\alpha$ should remain exact on S^2_∞. Therefore, for any smooth gauge choice we find

$$\int_{S^2_\infty} n^a d\mathcal{A}^a = 2\int_{S^2_\infty} n^a\mathcal{F}^a = 8\pi \ . \tag{4.7}$$

The potential subtlety is in the limiting case of the unitary gauge $n^a = \delta^{a3}$ where α is the Dirac potential of flux 4π with a Dirac string. Globally, $d\alpha$ remains exact. What happens here is that, in this gauge, the winding number density of the first term of $n^a\mathcal{F}^a$ is concentrated along the Dirac string direction and cancels the Dirac string contribution. For $n^a d\mathcal{A}^a$, this does not happen. Instead, the first, winding term overcompensate the Dirac string piece in $d\alpha$ by a factor of two. So the Dirac potential (i.e., $d\alpha$ minus the Dirac string) contributes 4π and the winding number density combined with the Dirac string contributes 4π, so that again we find $\int_{S^2_\infty} n^a d\mathcal{A}^a = 8\pi$.

Therefore, with $L = e^{\Lambda(x)\langle\Phi\rangle}$ and $\Lambda_* = \Lambda(\infty)$, the Euclidean action for a single monopole instanton is

$$-S_{\rm E} = 2 \times \frac{k}{4\pi}\int_{S^2_\infty} \Lambda_* \,\mathrm{tr}\,(\langle\Phi\rangle d\mathcal{A}) = 2 \times \frac{k\Lambda_*}{8\pi}\int_{S^2_\infty} n^a d\mathcal{A}^a = 2k\Lambda_* \tag{4.8}$$

which gives

$$e^{-S_{\rm E}} \; = \; e^{2k\Lambda_*} \; = \; \left(\frac{u_1}{u_2}\right)^k \quad \text{for} \quad \Lambda_* = \sqrt{\frac{u_1}{u_2}} \,. \tag{4.9}$$

The computation of the Euclidean action for the anti-monopole instanton proceeds exactly the same manner, except L is replaced by L^{-1} and $\mathcal{F} = \tilde{\mathcal{F}}$ has the opposite magnetic flux. The combined effect is again the same result. We could have done the same computation for multi-monopole instantons and multi-anti-monopole instantons, and the result is

$$e^{-S_{\rm E}} = \left(\frac{u_1}{u_2}\right)^{k|m|} \tag{4.10}$$

for the monopole number m. Note that our vacuum choice was such that $0 < u_1 < u_2$, and the WKB amplitude is suppressed by powers of $(u_1/u_2)^k$ for each monopole. This is consistent with $1/k$ as the effective coupling in this theory, for the amplitude is exponentially suppressed by k. However, the suppression is only powerlike with respect to the vacuum expectation values.

Now we turn to zero-mode counting. The number of bosonic zero-modes within the present ansatz with $\mathcal{A} = \tilde{\mathcal{A}}$ is clearly $4|m|$ since the problem collapses to the usual Yang-Mills case. While we do not have a rigorous proof yet, we believe these usual bosonic zero-modes of (anti-)BPS monopoles exhaust all such for the monopole instanton of the present theory. A partial support comes from the fermionic part of the story, which can be more easily counted. The fermionic partners, Ψ^α and $\bar{\Psi}_\alpha$ of Z_α and $\bar{Z}^\alpha$, have the following equation of motion when only $Z = Z_1$ is excited,

$$\frac{k}{2\pi}\gamma^\mu D_\mu \Psi^\alpha \pm (Z\bar{Z})\Psi^\alpha \mp \Psi^\alpha(\bar{Z}Z) = 0 \tag{4.11}$$

and

$$\frac{k}{2\pi}\gamma^\mu D_\mu \bar{\Psi}_\alpha \pm \bar{\Psi}_\alpha(Z\bar{Z}) \mp (\bar{Z}Z)\bar{\Psi}_\alpha = 0\,, \tag{4.12}$$

where again, in this Euclidean regime, we treat the two sets of fermions as independent. The upper sign is for $\Psi^{2,3,4}$ and $\bar{\Psi}_{2,3,4}$ while the lower sign is for Ψ^1 and $\bar{\Psi}_1$.

Let us first exploit the general form of monopole instanton solution, and go to $\mathcal{A} = \tilde{\mathcal{A}}, \mathcal{Z}, \bar{\mathcal{Z}}$ variables. Redefining

$$\Psi^\alpha = L\psi^\alpha L, \quad \bar{\Psi}_\alpha = L^{-1}\bar{\psi}_\alpha L^{-1}\,, \tag{4.13}$$

the zero-mode equations reduce to

$$\gamma^\mu \mathcal{D}_\mu \psi^\alpha \mp \mu[\Phi, \psi^\alpha] = 0 \tag{4.14}$$

and

$$\gamma^\mu \mathcal{D}_\mu \bar{\psi}_\alpha \pm \mu[\Phi, \bar{\psi}_\alpha] = 0\,, \tag{4.15}$$

where

$$\mathcal{D} = d - i\mathcal{A} \tag{4.16}$$

acting on what are effectively the adjoint fermions ψ and $\bar{\psi}$. The complication due to the complex nature of the solution does not enter the index counting because the scalar contributes only in terms of $\mathcal{Z}\bar{\mathcal{Z}} = -k\mu\Phi/2\pi + (\cdots) \times \mathbf{1}_2$. Note that the constant part $\mathcal{Z}\bar{\mathcal{Z}}$, proportional to $\mathbf{1}_2$, also disappears since the scalar $\mathcal{Z}\bar{\mathcal{Z}}$ acts as a commutator.

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Thus, the fermion zero-mode problem is reduced to that of 2-component adjoint fermions in ordinary BPS monopole, ($\mathcal{A} = \tilde{\mathcal{A}}, \mu\Phi$), albeit now in the Euclidean three-dimensional world. Since the monopole instanton is no longer a solution that obeys reality condition, the corresponding zero-mode counting could have been awkward. However, the special form of the solution $A, \tilde{A}, Z, \bar{Z}$ which can be mapped to $\mathcal{A} = \tilde{\mathcal{A}}, \mathcal{Z}, \bar{\mathcal{Z}}$, allows an easy translation to the zero-mode counting of the ordinary BPS monopole.

The latter says the following: the field equation for a complex fermion ψ in m-monopole background

$$\gamma^\mu \mathcal{D}_\mu \psi + \mu[\Phi, \psi] = 0 \tag{4.17}$$

has $2m$ zero-modes [40, 41], whereas the similar equation with the second term sign-flipped has no zero-modes. Thus on our one-monopole background we have two zero-modes from each of ψ_1, $\bar{\psi}_{2,3,4}$. The transforming matrix L does nothing to the usual normalizability conditions on zero-modes, so therefore we have total of eight zero-modes per each monopole instanton, with two each for

$$\Psi^1, \ \bar{\Psi}_2, \ \bar{\Psi}_3, \ \bar{\Psi}_4 \ . \tag{4.18}$$

For anti-monopoles, which also contribute quantum corrections, the situation is reversed and the roles of Ψ and $\bar{\Psi}$ are exchanged.

This apparent disparity between Ψ and $\bar{\Psi}$ is related to the usual practice of treating them as independent. What should be remembered, though, is that each zero-mode of Ψ, even though they are complex fields, carries a single fermionic collective coordinate and likewise for $\bar{\Psi}$. Thus, the number of Grassmanian collective coordinates to saturate, in order to have nonvanishing contribution to the path-integral, is eight. The vertex operators one can compute directly from the dilute gas approximation of monopoles and anti-monopoles should have eight fermions, of the form

$$(\Psi^1)^2 (\bar{\Psi}_2)^2 (\bar{\Psi}_3)^2 (\bar{\Psi}_4)^2 \ . \tag{4.19}$$

5. General monopole instantons, Euclidean action and zero-modes

So far we considered monopole instanton in a vacuum where only Z_1 takes an expectation value. Even in the simplest of the ABJM model with $\mathrm{U}(2) \times \mathrm{U}(2)$, however, this is not the generic vacuum. As we saw in section 2, generically three real parameters can be turned on, up to the gauge and $\mathrm{SU}(4)_\mathrm{R}$ symmetry transformations, and this forces at least two scalar fields, say $Z_{1,2}$, take vev as shown in eq. (2.18). In such general vacua, the ansatz we employed above will not work since the general form of the instanton solution requires turning on at least one more scalar field, say Z_2, in addition to $Z = Z_1$. In particular, the BPS equation has to be modified to accommodate Z_2 and $\bar{Z}^2$.

The generalized form of the BPS equation with two scalar fields involved is

$$DZ_1 = 0 \, , \qquad D\bar{Z}^2 = 0 \ . \tag{5.1}$$

This preserves one third of the $\mathcal{N} = 6$ supersymmetry with the preserved supersymmetry parameters η_{23}, η_{24} of the supersymmetry transformation (2.6). Note that a similar choice such as $DZ_1 = DZ_2 = 0$ would lead to the solutions with Z_1 and Z_2 proportional to each other, which are trivially related to the previous 1/2 BPS solutions by a suitable $\mathrm{SU}(4)_{\mathrm{R}}$ rotation. With this BPS equation, the Gauss constraints reduce to

$$\begin{aligned}\frac{k}{2\pi} * F &\equiv \frac{k}{2\pi} * (dA - iA \wedge A) = D(Z_2 \bar{Z}^2 - Z_1 \bar{Z}^1), \\ \frac{k}{2\pi} * \tilde{F} &\equiv \frac{k}{2\pi} * (d\tilde{A} - i\tilde{A} \wedge \tilde{A}) = D(\bar{Z}^2 Z_2 - \bar{Z}^1 Z_1)\,,\end{aligned} \tag{5.2}$$

which again suggests a simple mapping to ordinary monopole BPS equations, except that $\bar{Z}^2 Z_2 - \bar{Z}^1 Z_1$ replaces $-\bar{Z}Z$.

Recall that we chose the parameterization of the generic vacua (2.18) as

$$\langle Z_1 \rangle = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}, \qquad \langle Z_2 \rangle = \begin{pmatrix} cu_2 & 0 \\ 0 & cu_1 \end{pmatrix}, \tag{5.3}$$

where $0 < u_1 < u_2$ and $0 < c$. With this, we can again resort to the transformed variables

$$A = L\mathcal{A}L^{-1} + iLdL^{-1}, \quad \tilde{A} = L^{-1}\tilde{\mathcal{A}}L + iL^{-1}dL \tag{5.4}$$

and take the ansatz $\mathcal{A} = \tilde{\mathcal{A}}$. The Gauss constraints collapse to

$$*\mathcal{F} \equiv *(d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}) \;=\; \mu(d\Phi - i[\mathcal{A}, \Phi])\,, \tag{5.5}$$

where $\mu = \frac{2\pi}{k}(1+c^2)(u_2^2 - u_1^2)$ is the mass parameter for generic vacua (2.20). The monopole scalar function Φ (with $\mathrm{tr}\langle\Phi^2\rangle = 1/2$) enters the transformed scalar fields as

$$\begin{aligned}\mathcal{Z}_1 &= \sqrt{u_1 u_2}\,\mathbf{1}_2, \quad \bar{\mathcal{Z}}^1 = \frac{1}{\sqrt{u_1 u_2}}\left((u_1^2 - u_2^2)\Phi + \frac{u_1^2 + u_2^2}{2}\,\mathbf{1}_2\right), \\ \bar{\mathcal{Z}}^2 &= c\sqrt{u_1 u_2}\,\mathbf{1}_2, \quad \mathcal{Z}_2 = \frac{c}{\sqrt{u_1 u_2}}\left((u_2^2 - u_1^2)\Phi + \frac{u_2^2 + u_1^2}{2}\,\mathbf{1}_2\right),\end{aligned} \tag{5.6}$$

which is related to the physical scalar fields as,

$$Z_{1,2} = L\mathcal{Z}_{1,2}L, \quad \bar{Z}^{1,2} = L^{-1}\bar{\mathcal{Z}}^{1,2}L^{-1}, \tag{5.7}$$

for some L as before.

An interesting aspect of this solution is that L is independent of the constant c and remains unchanged from that of the monopole instanton in the special vacua. Thus L in equations (3.18) and (3.19) ensures the reality condition $\langle Z_{1,2} \rangle = \langle \bar{Z}^{1,2} \rangle^\dagger$. Because of this peculiar feature, which is no doubt due to our nonconventional parameterization of the vev's, the Euclidean action of the monopole instanton remains independent of c,

$$e^{-S_{\mathrm{E}}} = \left(\frac{u_1}{u_2}\right)^{k|m|} \tag{5.8}$$

for m-monopole instanton in this generic vacuum. We confirm this c-independent action from the M-theory calculation in section 7.

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As suggested by the fact that eight supercharges are broken, the number of zero-modes remains eight. Due to Z_1, Z_2 being nonzero, the fermion equation of motion mixes Ψ^1 and Ψ^2, and also Ψ^3 and $\bar{\Psi}_4$. Using that $\mathcal{Z}_1$ and $\bar{\mathcal{Z}}^2$ are constant and proportional to the identity matrix, we find that the equations for $\Psi^3, \Psi^4, \bar{\Psi}_3, \bar{\Psi}_4$ become

$$
\begin{aligned}
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \psi^3 + [\mathcal{Z}_1\bar{\mathcal{Z}}^1 + \mathcal{Z}_2\bar{\mathcal{Z}}^2, \psi^3] + 2[\mathcal{Z}_1\mathcal{Z}_2, \bar{\psi}_4] = 0, \\
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \psi^4 + [\mathcal{Z}_1\bar{\mathcal{Z}}^1 + \mathcal{Z}_2\bar{\mathcal{Z}}^2, \psi^4] - 2[\mathcal{Z}_1\mathcal{Z}_2, \bar{\psi}_3] = 0, \\
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \bar{\psi}_3 - [\mathcal{Z}_1\bar{\mathcal{Z}}^1 + \mathcal{Z}_2\bar{\mathcal{Z}}^2, \bar{\psi}_3] + 2[\bar{\mathcal{Z}}^1\bar{\mathcal{Z}}^2, \psi^4] = 0, \\
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \bar{\psi}^4 - [\mathcal{Z}_1\bar{\mathcal{Z}}^1 + \mathcal{Z}_2\bar{\mathcal{Z}}^2, \bar{\psi}^4] - 2[\bar{\mathcal{Z}}^1\bar{\mathcal{Z}}^2, \psi^3] = 0
\end{aligned} \tag{5.9}
$$

under $\Psi^\alpha = L\psi^\alpha L$ and $\bar{\Psi}_\alpha = L^{-1}\bar{\psi}_\alpha L^{-1}$.

Recalling $\mathcal{Z}_1\bar{\mathcal{Z}}^1 - \mathcal{Z}_2\bar{\mathcal{Z}}^2 = -k\mu\Phi/2\pi$ up to shifts by a constant multiple of identity matrix, we find that the following combinations

$$
\psi = \mathcal{Z}_1\bar{\psi}_3 - \bar{\mathcal{Z}}^2\psi^4 \quad \text{and} \quad \psi = \mathcal{Z}_1\bar{\psi}_4 + \bar{\mathcal{Z}}^2\psi^3
$$

satisfy the zero-mode equation (4.17). The other two linear combinations

$$
\begin{aligned}
\psi &= \sqrt{u_1u_2}\,(u_1^2 - u_2^2)\psi^3 + c\sqrt{u_1u_2}(u_2^2 - u_1^2)\bar{\psi}_4\,, \\
\psi &= \sqrt{u_1u_2}\,(u_1^2 - u_2^2)\psi^4 - c\sqrt{u_1u_2}(u_2^2 - u_1^2)\bar{\psi}_3
\end{aligned}
$$

satisfy the equation (4.17) with the second term sign-flipped, so that they do not yield zero-modes. The equations for $\Psi^1, \Psi^2, \bar{\Psi}_1, \bar{\Psi}_2$ read

$$
\begin{aligned}
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \psi^1 - [\mathcal{Z}_1\bar{\mathcal{Z}}^1 - \mathcal{Z}_2\bar{\mathcal{Z}}^2, \psi^1] = 2[\mathcal{Z}_2\bar{\mathcal{Z}}^1, \psi^2], \\
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \psi^2 + [\mathcal{Z}_1\bar{\mathcal{Z}}^1 - \mathcal{Z}_2\bar{\mathcal{Z}}^2, \psi^2] = 0, \\
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \bar{\psi}_1 + [\mathcal{Z}_1\bar{\mathcal{Z}}^1 - \mathcal{Z}_2\bar{\mathcal{Z}}^2, \bar{\psi}_1] = 0, \\
&\frac{k}{2\pi}\gamma^\mu \mathcal{D}_\mu \bar{\psi}^2 - [\mathcal{Z}_1\bar{\mathcal{Z}}^1 - \mathcal{Z}_2\bar{\mathcal{Z}}^2, \bar{\psi}^2] = -2[\bar{\mathcal{Z}}^1\mathcal{Z}_2, \bar{\psi}_1].
\end{aligned} \tag{5.10}
$$

The second and the third equations are (4.17) with the second term sign flipped, so they can only be solved by $\psi^2 = \bar{\psi}_1 = 0$. Inserting $(\psi^1, \psi^2) = (\psi, 0)$ or $(\bar{\psi}_1, \bar{\psi}_2) = (0, \psi)$ to the first or the fourth equations we get (4.17).

Summarizing, for each monopole instanton in generic vacuum, there are eight fermion zero-modes, with two each from

$$
\Psi^1, \quad \bar{\Psi}_2, \quad Z_1\bar{\Psi}_3 - \bar{Z}^2\Psi^4, \quad Z_1\bar{\Psi}_4 + \bar{Z}^2\Psi^3.
$$

This clearly reduces to the previous result for monopole instantons when $Z_2 = 0$. In this generic vacuum, the zero-modes of a single monopole are in one-to-one correspondence with the eight broken supercharges. Although the number of the broken supersymmetry is only six for monopoles in the special vacua, the number of zero-modes cannot change just by choice of the vacuum. The eight zero-modes per monopole therefore persist in all broken vacua, generic or special. This explains why we found eight zero-modes in the previous section, despite the half-BPS nature.

6. The vertex operator and non-perturbative effective action

The monopole instanton will contribute a local operator to the effective action. The purpose of this section is to discuss the possible form of such non-perturbative terms in the effective action. However, with the Chern-Simons term present, there is a subtlety one must first understand.

There is a well-known argument [34] that seemingly forbid the monopole instanton contribution to the Euclidean path integral for generic Chern-Simons theory. As a simple example, let us recall once again the SU(2) Chern-Simons theory with an adjoint scalar Φ. We discussed in section 4 how the Chern-Simons action transforms under complexified gauge transformations. Let us consider here the real gauge transformation of the form

$$g = e^{i\lambda\Phi}. \tag{6.1}$$

The scalar field is invariant under this, while the Euclidean action for m-monopole background is shifted by a pure imaginary constant,

$$\delta S_{\rm CS} = ikm\lambda, \tag{6.2}$$

where we used $\mathrm{tr}\,\langle\Phi\rangle^2 = \frac{1}{2}$. Now for $\lambda \not\in 2\pi\mathbb{Z}$, λ is neither a small gauge transformation nor a large gauge transformation, so the path-integral over all gauge field configuration implies integral over the gauge orbit, in other words an integral over λ from 0 to 2π. This seemingly projects out the contributions from the sectors with nonzero monopole number.

This argument, however, overlooks another important aspect of the Chern-Simons theory, where the Gauss constraint relates flux to electric charge. A monopole instanton induces a jump in total magnetic flux, and must be accompanied by a related jump in total electric charge. The final state and the initial state, mediated by the monopole instanton, differ by an U(1) electric charge $\sim km$. The constant gauge transformation by $\lambda(\infty)$ measures precisely this electric charge, so the product of wavefunctions also transform by a phase $e^{-ikm\lambda(\infty)}$. The transition amplitudes for monopole-mediated processes are therefore *not* projected out by integrating over the gauge orbit.

One can show the full gauge invariance of monopole-mediated amplitudes by taking account of the gauge variance of the Lagrangian carefully. To understand how to proceed, let us regard the system as a mechanical system with a dynamical variable $q(t)$ and the Lagrangian $L[q]$. Suppose the equation of motion is invariant under a group of symmetry transformations G, but L is invariant only up to total time derivative.

$$g \in G \;:\; q \longmapsto q^g\,, \quad L[q] \longmapsto L[q^g] = L[q] - \frac{d}{dt}2\pi\alpha_1[q,g]\,. \tag{6.3}$$

The functional α_1 is called *1-cocycle* due to the composition rule,

$$\alpha_1[q,g] + \alpha_1[q^g,g'] \;=\; \alpha_1[q,gg']. \tag{6.4}$$

The Noether charge gets modified due to this last term of (6.3), so that the corresponding quantum operator g acts on the basis states as follows,

$$\langle q|g = e^{2\pi i\alpha_1(q,g)}\langle q^g|\,. \tag{6.5}$$

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Now consider the transition amplitude between the states $|\Psi_i\rangle$ and $\langle\Psi_f|$ whose wave packets are localized near $q = q_i$ and $q = q_f$. The path integral gives

$$\left\langle \Psi_f | e^{-iH(t_f - t_i)} | \Psi_i \right\rangle \;=\; \int [dq] \Psi_f^*[q(t_f)] \Psi_i[q(t_i)] \exp\big(iS(t_f; t_i)\big) \,. \tag{6.6}$$

One can compute the kernel $\langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle$ approximately using the classical action for a stationary path connecting $q(t_i) = q_i$ and $q(t_f) = q_f$. The kernel is not invariant under G due to (6.3). Also, G-transformation of wave functions gives rise to a phase factor due to (6.5): the wave functions for the states $|\Psi\rangle$ and $|\Psi^g\rangle \equiv g|\Psi\rangle$ are related via

$$\Psi^g(q) = \Psi(q^g) e^{2\pi i \alpha_1(q,g)} \,. \tag{6.7}$$

The phase rotations of the kernel and wave functions cancel, so that the transition amplitudes are invariant. When applied to the previous Chern-Simons theory example and g is chosen to be a constant gauge transformation, these phase rotations reflect the flux of monopole instanton and the charges of the states.

The 1-cocycle α_1 is trivial if it is solved in terms a *0-cocycle* functional α_0,

$$\alpha_1[q, g] = \alpha_0[q^g] - \alpha_0[q], \tag{6.8}$$

since the theory is then described by a fully G-invariant Lagrangian

$$\tilde{L}[q] \;=\; L[q] + \frac{d}{dt} 2\pi\alpha_0[q] \,, \tag{6.9}$$

and the wave functions $\tilde{\Psi}(q) = \Psi(q) e^{2\pi i \alpha_0[q]}$ satisfying $\tilde{\Psi}^g(q) = \tilde{\Psi}(q^g)$. However, in Chern-Simons theories the 1-cocycle can only formally be solved, and the resulting 0-cocycle turns out to be a nonlocal functional [42, 43]. In appendix A we record an explicit form of α_0 for a simple BF-matter theory.

Let us turn to discuss in some detail the gauge transformation property of our monopole solution in the $U(2) \times U(2)$ ABJM model. In sections 3 and 5 we solved the equations of motion by a simple embedding of the 't Hooft Polyakov monopole $(\mathcal{A}, \Phi)$. The embedding is such that the classical Chern-Simons action for the two $U(2)$ gauge fields cancel, but the scalars $\mathcal{Z}_\alpha$ and $\bar{\mathcal{Z}}^\alpha$ are not conjugate of each other at infinity. A complexified gauge transformation can correct this wrong asymptotics, but makes the total Chern-Simons action non-vanishing. The end result was $e^{-S_{\rm E}} = (u_1/u_2)^k$ for one monopole where u_1, u_2 are the eigenvalues of $\langle Z_1 \rangle$. Speaking in terms of cocycles, what we have done is to use the 1-cocycle relation (6.3) to relate the values of classical action in a "wrong gauge" to a "real gauge".

The Euclidean action is therefore not invariant under some gauge transformations. Indeed, the vevs of Z_α are simultaneously diagonalizable and in general break the gauge group from $U(2) \times U(2)$ down to $U(1)^4$. A $U(1)^2$ subgroup rotates u_1 and u_2 by independent phases, and shifts the Euclidean action by pure imaginary constant. The monopoles carry charges under the $U(1)$ group which phase-rotates u_1 and u_2 oppositely.

However, this does not imply the monopole effect is projected out, because we are not integrating over this gauge orbit. As was reviewed in section 2, the moduli space of vacua is

$(\mathbb{C}^4/\mathbb{Z}_k)^2/S_2$, and in particular the two vacua labelled by (u_1, u_2) and $(e^{-i\lambda_1/k}u_1, e^{-i\lambda_2/k}u_2)$ are not gauge equivalent unless $\lambda_i \in 2\pi\mathbb{Z}$. This is precisely because of the monopole-instantons breaking $\mathrm{U}(1)^2$ down to $(\mathbb{Z}_k)^2$. Our monopole-instanton action is clearly invariant under this orbifold group, and it can be lifted to a well-defined function on the moduli space.

Thus we can find the instanton contribution to the effective action, weighted by $e^{-S_{\mathrm{E}}}$. As emphasized before, the monopole-instanton carries the electric charges in addition to the creation or annihilation of certain magnetic flux. We therefore consider the charge-flux creation operator, or simply vertex operator, to describe the effective interactions induced by those instantons. The charge creation operators are in general non-local operators because of their long-range electric fields. In the Chern-Simons theories, however, the electrically charged states do not emit the electric field, but are tied with local magnetic flux. It implies that the charge-flux creation operators can now become local. For example, local gauge-invariant charge-flux creation operators for scalar fields z^i_α are given by $\tilde{z}^i_\alpha$ (2.15),

$$\tilde{z}^1_\alpha = e^{-i\theta/k + i\sigma/2k} z^1_\alpha, \qquad \tilde{z}^2_\alpha = e^{-i\theta/k - i\sigma/2k} z^2_\alpha\,, \tag{6.10}$$

where $\theta = \frac{1}{2}(\theta^1 + \theta^2)$ and $\sigma = \theta^2 - \theta^1$. It is the σ normalized to have period 2π that properly describes the effect of the monopole-instantons. Some details are explained in appendix C.

The vertex for the instanton has to do two things. First it should create or destroy certain quantized magnetic flux, which can be written in terms of a dual photon field σ as $e^{im\sigma}$. In appendix C, we show that this is indeed the case. Thus the rough form of the gauge-invariant vertex is $(m > 0)$

$$e^{-S_{\mathrm{E}} + im\sigma} = \left(\frac{z^1}{z^2}\right)^{km} e^{im\sigma} = \left(\frac{\tilde{z}^1}{\tilde{z}^2}\right)^{km} \tag{6.11}$$

since our notation is such that $\langle z^i \rangle = u_i$. Second, the vertex must also carry km units of an electric charge. For $m > 0$, the vertex we wrote already reflects this since z^1_α and z^2_α are oppositely charged at unit $\pm 1/2$.

Incorporating the effect of fermionic zero-modes and the conformal invariance, we expect further prefactors from zero-modes and massive modes. The net effect is to have additional nonperturbative corrections to the effective Lagrangian in the broken phase,

$$\mathcal{L}_{\text{non-perturbative}} = \sum_m \left(g_{k,m}(\tilde{z}, \bar{\tilde{z}}, \nabla\tilde{z}, \nabla\bar{\tilde{z}}, \tilde{\Psi}, \bar{\tilde{\Psi}}) \left(\frac{z_1}{z_2}\right)^{km} e^{im\sigma} + c.c \right) \tag{6.12}$$

where $g_{k,m}$ are dimension-three and charge-neutral operators. When we do not consider motion of the vacuum moduli ($\nabla\tilde{z} = 0 = \nabla\bar{\tilde{z}}$), the only possible term is the eight-fermion term with

$$g_{k,m} \sim \frac{f_8(\tilde{\Psi}, \bar{\tilde{\Psi}}, \tilde{z}, \bar{\tilde{z}})}{\mu^5(\tilde{z}, \bar{\tilde{z}})}\,, \tag{6.13}$$

where $\mu(\tilde{z}, \bar{\tilde{z}}) = \mu(z, \bar{z})$ denotes the unique mass parameter (2.20) on the vacuum moduli space and f_8 is an 8-th order polynomial in the fermions with dependence on the scalar vev

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only through ratios. The charge-neutrality here implies that $f_8(\tilde{\Psi}, \bar{\tilde{\Psi}}, \tilde{z}, \bar{\tilde{z}}) = f_8(\Psi, \bar{\Psi}, z, \bar{z})$. Here we indicated only the rough scaling behavior. One can further restrict the possible structure of this term from the non-anomalous $\mathrm{SU}(4)_{\mathrm{R}}$ symmetry.

If we allow motion of the vacuum moduli, we will have various mixing terms between fermions and ∇z, $\nabla \bar{z}$. Recalling the discussion in [30] about eight fermion zero-modes, we believe that the purely bosonic terms generated by instanton effects should start with four-derivatives

$$g_{k,m} \sim \frac{|\nabla \tilde{z}|^4}{\mu^3(\tilde{z}, \bar{\tilde{z}})} \tag{6.14}$$

again up to a dimensionless neutral operator. Determining the structure of these vertex operators in full detail is beyond the scope of this work and needs more careful analysis. In the next section, we will try to compare the four-derivative terms (6.14) to those in the dual supergravity picture.

7. M/IIA bulk computation

The $\mathrm{U}(N) \times \mathrm{U}(N)$ ABJM model is believed to be the worldvolume theory of N M2-branes in $\mathbb{C}^4/\mathbb{Z}_k$ orbifold. This proposal is so far supported by several basic evidences. One is that the theory has the right supersymmetry and conformal symmetry. Another is that the massless degrees of freedom of the ABJM model match precisely with those of the nonlinear sigma models of such M2-branes. Also it has been shown that counting of superconformal indices [19] is consistent with this proposal.

However, there is also a potentially contradicting piece of evidence that seems to say that the number of isolated vacua of a mass-deformed ABJM model is different from what is expected from the bulk side in the large N limit [17]. Given these mixed results, it is natural to ask whether we can find further supporting evidences by considering more sophisticated aspects of the theory, such as quantum-corrected interactions.

An interesting analog can be found by considering D2-branes on flat $\mathbb{R}^7$, which are nothing but M2-branes on $S^1 \times \mathbb{R}^7$. The worldvolume theory of multiple D2-branes is given by $\mathcal{N} = 8$ $\mathrm{U}(N)$ Yang-Mills theory, where monopole instantons of usual kind exist in the Coulomb phase where an adjoint scalar ϕ takes a vev. Polchinski and Pouliot [30] computed, for the case of U(2), what kind of interactions are generated by these instantons and found four-derivative terms, such as $e^{-4\pi\phi/e^2}(\nabla\phi)^4/\phi^3$, and its supermultiplet up to eight fermion terms, suppressed exponentially by the Euclidean action of the instanton.

On the other hand, since D2-branes are really M2-branes, a pair of D2-branes separated by a distance r in the IIA theory exchange 11-dimensional supergravitons. In particular, when the momenta being exchanged are those associated with the M-theory circle S^1, this generates quantum correction of type $e^{-mr/R}(\nabla r)^4/r^3$ where R is the radius of the eleventh circle and m is a positive integer. Alternatively we can think of this process as exchange of m D0-branes. Since we can interpret the worldvolume quantities as $\alpha'\phi \sim r$ and $e^2\alpha' \sim R$, this interaction term computed from M-theory is exactly the same as the four-derivative monopole instanton vertex above computed from $\mathcal{N} = 8$ Yang-Mills theory.

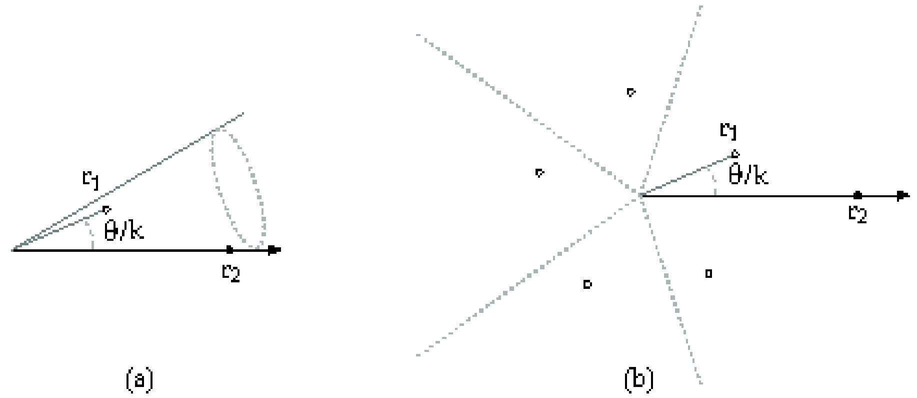

Figure 1: (a) Two M2-branes placed in the $\mathbb{C}/\mathbb{Z}_k$ subspace of the cone $\mathbb{C}^4/\mathbb{Z}_k$. (b) The covering space view of the same configuration.

Here we would like to make a similar comparison for the ABJM proposal of M2-branes. In previous sections, we already discussed how the monopole instantons lead to quantum correction to the effective Lagrangian at the level of four-derivative terms and the supermultiplet thereof. Although we did not derive the exact form of the vertex, we did derive the leading k-dependence of the vertex and also how it scales with the mass scale μ of the generic Coulombic vacua. In the following, we will compare these four-derivative vertices to those found in the bulk computation where M2-branes scatter off each other via M-theory supergraviton exchange or alternatively where D2-branes interact via exchange of D0-branes.

7.1 M-theory picture: four-derivative interactions

We think of the two M2-branes as a source and a probe. The source produces a background field configuration,

$$ds^2 = h^{-2/3} dx_{1+2}^2 + h^{1/3} dy^2 \,, \qquad C_{012} = h^{-1} \,, \tag{7.1}$$

where the harmonic function for a single M2-brane is given by

$$h = 1 + \frac{32\pi^2}{(M_{11} r)^6} \,. \tag{7.2}$$

Before proceeding further, however, we wish to argue that the right thing to do to make a comparison against the gauge theory result is to drop "1" in the harmonic function.

One way to achieve this naturally is to consider the number of "source" M2-branes to be very large and take the near horizon limit. On the field theory side, this amounts to considering $\mathrm{U}(N+1)$ theory broken to $\mathrm{U}(N) \times \mathrm{U}(1)$. The latter would involve further complication due to the fact that the monopole instanton carries $\mathrm{U}(N)$ charge, which we would like to avoid.

Another is to compute everything as it is and then extrapolate to small r regime, while maintaining the velocity of M2-branes also sufficiently small. A priori, there is no overlap between the regime where this bulk computation is trustable (i.e., long distance regime) and the regime where the worldvolume gauge theory computation is reliable (i.e., short distance regime). Nevertheless, with enough supersymmetry, the structure of interactions mediated by BPS objects tends to be preserved across such interpolations. This has been seen time and again in the development of D-brane physics. We will be testing the ABJM proposal against the bulk computation, in this sense. Performing such an extrapolation carefully is equivalent to using

$$h = \frac{32\pi^2}{(M_{11}r)^6} \tag{7.3}$$

from the very start.

We then study the dynamics of the probe brane using

$$S_{\text{probe}} = T_2 \int d^3x \left(-\sqrt{-g} + \frac{1}{6}\epsilon^{\mu\nu\lambda}\partial_\mu X^M \partial_\nu X^N \partial_\lambda X^P C_{MNP} \right) , \tag{7.4}$$

where $g_{\mu\nu}$ here is the pull-back of the metric (7.1). We will focus on a slow motion in a direction transverse to the M2-brane worldvolume and perpendicular to the separation between the two branes, following a similar computation in flat background [32, 33].

From this, we anticipate to reproduce four-derivative terms such as in eq. (6.14). Given the 2+1-dimensional Lorentz invariance, however, it suffices to consider uniform motion of the M2-branes, encoded in velocities $v = \partial_t X$, instead of considering ∇X. Expanding the probe action in powers of velocity v, we find that the (v^0) term vanishes due to the BPS cancellation between the two terms. The (v^2) term serves as the kinetic term and the (v^4) term is the leading interaction term. Explicitly, the action up to the (v^4) term is given by[3]

$$S_{\text{probe}} = \int d^3x \left[\frac{1}{2} T_2 v^2 + \frac{1}{8} T_2 h v^4 + \mathcal{O}(v^6) \right] . \tag{7.5}$$

Suppose the two M2-branes are located at $\vec{z}$ and $\vec{w}$ in $\mathbb{C}^4$. Without loss of generality, we may assume $|\vec{w}| > |\vec{z}|$, and define

$$x \equiv \frac{|\vec{z}|}{|\vec{w}|} < 1, \qquad y e^{i\sigma/k} \equiv \frac{\vec{z}^* \cdot \vec{w}}{|\vec{z}||\vec{w}|} \quad (0 \le y \le 1, 0 \le \sigma \le 2\pi k) . \tag{7.6}$$

For a later comparison with the field theory computation, it is convenient to use the rescaled field theory variables

$$Z_a^{\text{F.T.}} = \sqrt{\frac{T_2}{2}} \left(X_{2a-1} + i X_{2a} \right)^{\text{Grav}} = \frac{M_{11}^{3/2}}{2\sqrt{2\pi}} \left(X_{2a-1} + i X_{2a} \right)^{\text{Grav}} . \quad (a = 1, \cdots, 4) \tag{7.7}$$

From now on we mean by z and w these rescaled coordinates of dimension 1/2. The velocity v is rescaled by the same factor to become a variable of dimension 3/2.

The $\mathbb{Z}_k$ orbifolding introduces mirror images of $\vec{z}$ at $e^{2\pi i\ell/k}\vec{z}$ ($\ell = 1, \cdots k$), so instead of having a single hv^4 term, we will have k copies of h with rotated centers contributing.

[3] The intermediate step goes like: $-h^{-1}\sqrt{1 - hv^2} + h^{-1} = \frac{1}{2}v^2 + \frac{1}{8}hv^4 + \mathcal{O}(v^6)$.

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This effectively replaces h by (up to an overall normalization),

$$F_k(\vec{z},\vec{w}) \equiv \sum_{\ell=1}^{k} |\vec{w} - e^{2\pi i l/k}\vec{z}|^{-6}$$
$$= \sum_{\ell=1}^{k} \frac{1}{\left(|\vec{z}|^2 + |\vec{w}|^2 - 2|\vec{z}^* \cdot \vec{w}|\cos(2\pi\ell/k + \sigma/k)\right)^3}\,, \tag{7.8}$$

which reduces the periodicity of the harmonic function to 2π. The angular coordinate σ is to be identified with the dual photon field that makes appearance in the monopole instanton vertex, and the m-instanton amplitude is expected to be proportional to the m-th Fourier coefficients of $F_k(\vec{z},\vec{w})$;

$$F_k(\sigma) = \sum_{m=-\infty}^{\infty} f_{k,m} e^{im\sigma}\ . \tag{7.9}$$

Each and every summand represents the monopole vertex of type (6.14). This is an expansion of the four-derivative interaction between a pair of the M2-branes, in terms of the angular momentum m of the angle σ. In type IIA interpretation, as we will see later, m labels the number of D0-branes being exchanged by the pair of D2-branes. D0-brane is still the Kaluza-Klein momentum of the 11-th direction, although the latter is now an azimuthal angle rather than a topological circle. Collecting the results, we find the following effective action in terms of the field theory variables

$$S_{\text{probe}} = \int d^3x \left[v^2 + \frac{v^4}{8\pi^2}\Big(f_{k,0}(\vec{z},\vec{w}) + \sum_{m=1}^{\infty} f_{k,m}(\vec{z},\vec{w})(e^{im\sigma} + e^{-im\sigma})\Big)\right]\,, \tag{7.10}$$

up to order v^6.

We thus find the M-theory counterpart of (6.14) as

$$\frac{v^4}{8\pi^2} f_{k,m} e^{im\sigma}\,, \qquad f_{k,m} = \int_0^{2\pi} \frac{d\sigma'}{2\pi}\, F_k(\sigma') e^{-im\sigma'}\,, \tag{7.11}$$

where the overall normalization is fixed by combining (7.3) and (7.5) and taking the rescaling (7.7) into account. We can combine the σ'-integral and the sum over mirror images into an integral over the circle in the "covering space" ($\sigma'/k \to \beta$),

$$f_{k,m}(\vec{z},\vec{w}) = k \int_0^{2\pi} \frac{d\beta}{2\pi} \frac{e^{-imk\beta}}{\left(|\vec{z}|^2 + |\vec{w}|^2 - 2|\vec{z}^* \cdot \vec{w}|\cos\beta\right)^3}\,. \tag{7.12}$$

The integral can be most easily evaluated by a contour integral along a unit circle on the complex plane ($e^{i\beta} \to z$). The result is

$$f_{k,m}(\vec{z},\vec{w}) = \frac{8\pi^2 q^{k|m|}}{(2\pi/k)^3 (q^{-1} - q)^3 |\vec{z}^* \cdot \vec{w}|^3} \cdot a_{k,m}(q)\,, \tag{7.13}$$

where $q < 1$ is defined by

$$q + \frac{1}{q} = \frac{1}{y}\left(x + \frac{1}{x}\right)\,, \tag{7.14}$$

and

$$a_{k,m}(q) = \frac{\pi m^2}{2} + \frac{3\pi|m|(1+q^2)}{2|k|(1-q^2)} + \frac{2\pi(1+4q^2+q^4)}{k^2(1-q^2)^2}\,. \tag{7.15}$$

The match with the field theory counterpart is easily seen by noting that the parametrization of the generic vacuum (2.18) translates to

$$\vec{z} = (u_1, cu_2, 0, 0), \quad \vec{w} = (u_2, cu_1, 0, 0). \tag{7.16}$$

With this choice, the relation (7.14) yields $q = u_1/u_2$. So one can identify the suppression factor (exponential in k) as the Euclidean action

$$e^{-S_{\mathrm{E}}} = q^{k|m|}\,, \tag{7.17}$$

which matches precisely with the field theory analysis (5.8). Furthermore, the dependence on the fundamental scale is also reproduced correctly since

$$(2\pi/k)^3(q^{-1}-q)^3|\vec{z}^* \cdot \vec{w}|^3 = \left(\frac{2\pi}{k}(1+c^2)(u_2^2-u_1^2)\right)^3 = \mu^3\,. \tag{7.18}$$

Interestingly, the dependence on the variable c appears only through this mass scale term in (7.13). Thus the transfer of m unit of momenta along σ direction generates the following term in the probe M2-brane dynamics

$$\frac{v^4}{8\pi^2} f_{k,m}(\vec{z},\vec{w})e^{im\sigma} = \frac{v^4 q^{k|m|}e^{im\sigma}}{\mu^3}\, a_{k,m}(q)\,. \tag{7.19}$$

This is consistent with the monopole instanton vertex in eq. (6.14). Thus, we find that the ABJM field theory at the nonperturbative level captures the behavior of multiple M2 brane physics faithfully.

The field theoretical computation can be further improved. For instance, the above M-theory computation provides the exact expression for the prefactor in the form of $a_{k,m}(q)$ in (7.15), which captures the complicated dependence on ratios of the vev. This, together with $1/\mu^3$ factor, should match the higher order corrections to the saddle point approximation in the field theory side. Also $m = 0$ term in the effective Lagrangian, corresponding to the supergraviton exchange in the sector where σ momentum is zero, should come from ordinary perturbative corrections in the field theory side. More precise check of the ABJM proposal should be possible by computing these two classes of quantum corrections.

7.2 Consistency check with IIA picture

D0-brane probe in $\mathbb{C}^4/\mathbb{Z}_k$. The bulk picture can be thought of in two equivalent ways. In the above M-theory picture, we have N M2-branes in the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. In the second, related picture, we have N D2-branes in $\mathbb{CP}^3$ with nonconstant 11-th radius and a nontrivial RR field strength dC_1. In the latter, the series of interactions we found above can be understood as exchange of different number of D0-branes between a pair of D2-branes. Here, we will work in this latter picture and work out the k and m dependence of the amplitude according to the D0-brane exchange picture.

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Since we have no compact S^1, one might wonder what D0-branes are from the M-theory perspective. Note that the above expansion of the M-theory effective Lagrangian to sectors with different m is nothing but expansion of the full 11-dimensional amplitude into some angular-momentum eigensectors. If we choose to label the associated angle as the 11-th direction, the quanta of its conjugate momentum should be called D0-branes. Even though this 11-th direction does not define a topological circle, it is still a Killing direction so that we have a conserved conjugate momentum. IIA picture will see these quanta as D0-branes, Here we wish to confirm whether the individual amplitudes are consistent with the interpretation in terms of the D0-brane worldline viewpoint.

We work in the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold which is the vacuum moduli space of the ABJM model. The metric is given by

$$ds_{\rm M}^2 = dx_{1+2}^2 + dr^2 + r^2\left\{d\Omega_{\mathbb{CP}^3}^2 + \frac{1}{k^2}(d\psi + k\,C)^2\right\}. \tag{7.20}$$

We rescaled the angle of the S^1 fiber such that ψ has period 2π. The 1-form C satisfies $dC = 2J$ where J is the standard Kähler form of $\mathbb{CP}^3$. KK reduction along the ψ direction gives the IIA background,

$$\begin{aligned} ds_{\rm IIA}^2 &= \left(\frac{r}{k}\right)\left[dx_{1+2}^2 + dr^2 + r^2 d\Omega_{\mathbb{CP}^3}^2\right], \\ e^{2\phi} &= \left(\frac{r}{k}\right)^3, \qquad\qquad C_{(1)} = k\,C. \end{aligned} \tag{7.21}$$

Now, imagine placing two M2-branes in the geometry. We use the probe approximation, that is, we neglect the back-reaction to the geometry. Let $\vec{z}, \vec{w} \in \mathbb{C}^4$ be the coordinates of the two M2-branes in the covering space.

In the IIA picture, the instanton in question is a Euclidean D0-brane connecting the two D2-branes. The dynamics of the D0-brane should be captured by $e^{-S_{\rm DBI}+iS_{RR}}$, where

$$\begin{aligned} S_{\rm DBI} &= \int e^{-\phi} d\ell \;=\; k\int\sqrt{(dr/r)^2 + ds_{\mathbb{CP}^3}^2}, \\ S_{\rm RR} &= \int C_{(1)} = k\int C. \end{aligned} \tag{7.22}$$

For simplicity, let us first focus on the simple case where the two D2-branes are located on the same point in $\mathbb{CP}^3$, but separated in the r-direction; take $c = 0$ in equation (7.16). Then, we find

$$e^{-|m|S_{\rm DBI}} = \left(\frac{u_1}{u_2}\right)^{k|m|}, \tag{7.23}$$

which again coincides with the field theory result.

In general, with separation in $\mathbb{CP}^3$, the problem gets more complicated due to the presence of the RR-coupling (7.22). Let us sketch how a similar analysis goes through in this case. We first notice that one can always move the two M2-branes to lie in $\mathbb{C}^2 \subset \mathbb{C}^4$ by using the SU(4) rotation. Using the standard coordinates,

$$(z_1, z_2) = re^{i\psi}\Big(\cos(\theta/2)e^{i\phi/2}, \sin(\theta/2)e^{-i\phi/2}\Big), \tag{7.24}$$

and dimensionally reducing along ψ we get to the IIA picture. The Euclidean D0-brane has the worldline action $S = ks$, where

$$s = \frac{1}{2}\int\left(\sqrt{dt^2 + d\theta^2 + \sin^2\theta d\phi^2} - i\cos\theta d\phi\right), \qquad \big(t \equiv \log(r^2)\big). \tag{7.25}$$

The classical variational problem becomes well defined once we Wick rotate the variable $\phi = i\varphi$ to make the action real. The problem is to find the stationary path connecting two points $(t_i, \theta_i, \varphi_i)$ and $(t_f, \theta_f, \varphi_f)$ with $\varphi_i = \varphi_f = 0$. Using the explicit solution to the equation of motion, one can show the classical action satisfies

$$\cos((\theta_i - \theta_f)/2)\cosh s = \cosh((t_f - t_i)/2), \tag{7.26}$$

which is in precise agreement with (7.14).

Mass of fundamental string. Another important part of the four-derivative vertices is $1/\mu^3$ piece, which is determined by two considerations. First, $1/\mu^3$ carries the right dimension to render vertices to be of dimension three, making the interaction conformal. Second, the massive particles in the Coulombic vacua is set by the unique fundamental scale μ, so its appearance is natural. For the M2-brane interpretation of the ABJM model to make sense, μ should correspond to the mass of an open M2-brane wrapping M-theory circle and stretching between the two M2-branes, or that of a fundamental string stretched between the pair of D2-branes. While this is an easy task, we show it here since we chose a rather unconventional parameterization of the vev in the field theory.

The mass of a fundamental string stretched between two D2-branes is given by[4]

$$\mu_{\text{bulk}} = T_2\left(\frac{2\pi}{k}\right)\int\sqrt{r^2dr^2 + r^4d\theta^2}\,. \tag{7.27}$$

The curve that minimizes the mass is found to be

$$r^2(\theta) = \frac{2a}{\sin 2(\theta + \theta_0)}\,, \tag{7.28}$$

where a and θ_0 are constants. Using the boundary values (7.16),

$$\begin{aligned}(T_2/2)\,r^2(\theta_1) &= u_1^2 + c^2u_2^2, \qquad \tan\theta_1 = \frac{cu_2}{u_1},\\ (T_2/2)\,r^2(\theta_2) &= u_2^2 + c^2u_1^2, \qquad \tan\theta_2 = \frac{cu_1}{u_2},\end{aligned} \tag{7.29}$$

we can determine a and θ_0,

$$T_2\,a = 2cu_1u_2, \qquad \theta_0 = 0\,. \tag{7.30}$$

[4] Here we are using the standard polar coordinates for $\mathbb{R}^2$. The coordinate θ here is different from that in (7.24).

Inserting them back into the mass functional (7.27), we obtain

$$\begin{aligned}\mu_{\rm bulk} &= T_2\left(\frac{2\pi}{k}\right)\int_{\theta_2}^{\theta_1} d\theta\,\frac{2a}{\sin^2 2\theta} \\ &= T_2\left(\frac{2\pi}{k}\right)\times a\left[\frac{\cos 2\theta_2}{\sin 2\theta_2}-\frac{\cos 2\theta_1}{\sin 2\theta_1}\right] \\ &= \left(\frac{2\pi}{k}\right)(1+c^2)(u_2^2-u_1^2)\,, \end{aligned} \tag{7.31}$$

in perfect agreement of the mass scale μ in the broken phase of the field theory (2.20).

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A. Cocycles in a BF theory

In this section we illustrate how the gauge variance of the Lagrangian can be improved by adding the 0-cocycle, and how to obtain it by solving the Gauss constraint. As a simple example, we consider the abelian BF-matter theory which arises in the low-energy effective theory of the ABJM model.

It is important that the Lagrangian for Chern-Simons theories is first order in time derivative. The spatial components of the gauge fields are therefore divided into canonical coordinates and momenta by a choice of polarization, whereas the time components are Lagrange multipliers for the Gauss constraint. The cocycle then depends also on the polarization, recalling that the first order Lagrangian $L = p\dot{q} - H(p,q)$ transform under the canonical transformation $(q,p)\to(p,-q)$ as

$$L' - L \;=\; -q\dot{p} - p\dot{q} \;=\; -\frac{d}{dt}(pq)\,. \tag{A.1}$$

Let us consider the BF-matter theory with the Lagrangian

$$\mathcal{L} = -|D_\mu z|^2 + \frac{k}{4\pi}\Big(b_0(\partial_1 c_2 - \partial_2 c_1) + c_0(\partial_1 b_2 - \partial_2 b_1) + b_2\dot{c}_1 + c_2\dot{b}_1\Big). \tag{A.2}$$

The canonical coordinates are z, b_1, c_1, and the commutation relation in the temporal gauge reads

$$[c_i(\mathbf{x}), b_j(\mathbf{y})]_{\rm ET} \;=\; \frac{4\pi i}{k}\epsilon_{ij}\delta^2(\mathbf{x}-\mathbf{y}). \tag{A.3}$$

The physical wave function $\Phi(z, b_1, c_1)$ satisfies the Gauss constraints,

$$\begin{aligned}\left(i\frac{\delta}{\delta\theta(\mathbf{x})} - i\partial_1\frac{\delta}{\delta b_1(\mathbf{x})} - \frac{k}{4\pi}\partial_2 c_1(\mathbf{x})\right)\Phi &= 0,\\ \left(-i\partial_1\frac{\delta}{\delta c_1(\mathbf{x})} - \frac{k}{4\pi}\partial_2 b_1(\mathbf{x})\right)\Phi &= 0,\end{aligned} \tag{A.4}$$

where θ is the canonical conjugate of the gauge charge density. The solution is

$$\Phi \;=\; \exp\left\{\frac{ik}{4\pi}\int d^2\mathbf{x}\, c_1(\mathbf{x})\partial_1^{-1}\partial_2 b_1(\mathbf{x})\right\}\tilde{\Phi}(z(\mathbf{x})e^{-i\partial_1^{-1}b_1(\mathbf{x})}). \tag{A.5}$$

The exponential part is identified as the cocycle,

$$2\pi\alpha_0(b_1, c_1) = -\frac{k}{4\pi}\int d^2\mathbf{x} c_1(\mathbf{x})\partial_1^{-1}\partial_2 b_1(\mathbf{x}). \tag{A.6}$$

Under the local gauge transformations, the action $S = \int_{t_i}^{t_f} dt\mathcal{L}$ is not invariant, but can be made invariant by adding the boundary terms from cocyles,

$$\mathcal{S}_{\text{inv}} \;\equiv\; \int_{t_i}^{t_f} dt d^2\mathbf{x}\,\mathcal{L} \;+ 2\pi\alpha_0(b_1, c_1, t_f) - 2\pi\alpha_0(b_1, c_1, t_i). \tag{A.7}$$

The remaining part of the wave function $\tilde{\Phi}(z(\mathbf{x})e^{-i\partial_1^{-1}b_1(\mathbf{x})})$ is invariant under local gauge transformation. For states with charge n, $\tilde{\Phi}$ is a homogeneous function of order n. The monopole action could have contributions from both S_{inv} and $\tilde{\Phi}$, as one can see in the appendix B.

B. Complex action and monopole action

To acquaint the complex action and its stationary path, let us consider a simple mechanics model with a rotational symmetry. With the periodic coordinate $\theta \sim \theta + 2\pi$, its Lagrangian and Hamiltonian are $L = r^2\dot{\theta}^2/2$ and $H = p^2/2r^2$, respectively, where p is the conserved angular momentum. We are interested in calculating the amplitude

$$W = \frac{\langle\Psi_f|e^{-HT}|\Psi_i\rangle}{\langle\Psi_f|\Psi_f\rangle^{\frac{1}{2}}\langle\Psi_i|\Psi_i\rangle^{\frac{1}{2}}} \tag{B.1}$$

between initial and final states of angular momentum p_f, p_i. We choose the wave functions to be functions of coordinate so that $\Psi_i \sim e^{ip_i\theta}$. The norm of the initial and final wave functions are not relevant. One can express the above amplitude as a path integral

$$\int [d\theta]\Psi(\theta_f)^* e^{-S_{\text{E}}}\Psi(\theta_i) = \int [dpd\theta]e^{-S_{\text{E}}-S_{\text{b}}}\,, \tag{B.2}$$

where the Euclidean action and the boundary contribution are given by

$$S_{\text{E}} = \int d\tau\left(-ip\dot{\theta} + \frac{p^2}{2r^2}\right)\,, \qquad S_{\text{b}} = i(p_f\theta_f - p_i\theta_i)\,. \tag{B.3}$$

It is easy to find the stationary path of the above path integral. From the p, θ variations, we get $p = ir^2\dot{\theta}$, $\dot{p} = 0$ and $p(t_{f,i}) = p_{f,i}$. Note that the boundary variations of $\theta_{f,i}$ fix the initial and the final momenta. The solution is that $p_f = p_i$ and $\theta = -ip_i\tau/r^2$ up to a constant shift of τ. The total action becomes

$$S_{\rm E} + S_{\rm b} = +\frac{p_i^2 T}{2r^2} \,. \tag{B.4}$$

This is exactly what we expect from an energy eigenstate of $E = p_i^2/2r^2$. As $S_{\rm E} = -p_i^2/2r^2$, the wave function contribution is crucial. Note that the stationary path of angle has an imaginary direction. One point is that the phase is purely imaginary at the stationary point and so that $e^{i\theta}$ and $e^{-i\theta}$ are not complex conjugate to each other along the stationary path.

We are applying the similar idea for our monopole instantons. The partition function Z can be written as

$$W = \int [d\phi] \Psi_f(z^i)^* e^{-S_{\rm E}} \Psi_i(z^i) \,. \tag{B.5}$$

The monopole instanton is interpolating two states whose charge difference is km and so the vacuum wave function on S^2_∞ is

$$\Psi_f(z^i)^* \Psi_i(z^i) \sim \left(\frac{z_1}{z_2}\right)^{n_1} \left(\frac{\bar{z}_1}{\bar{z}_2}\right)^{n_2} \,. \tag{B.6}$$

We consider here only spatially homogeneous mode of the fields. This carries zero charge under $z^i \to e^{i\epsilon} z^i$ and carries km charge under $z^1 \to e^{i\lambda} z^1, z^2 \to e^{-i\lambda} z^2$ if

$$n_1 - n_2 = km \,. \tag{B.7}$$

In terms of the phase θ_i of the z^i fields, the wave function becomes

$$\langle \Psi_f | \Psi_i \rangle \sim e^{km(i\theta_1 - i\theta_2)} \,. \tag{B.8}$$

The modulus of the wave function cancels and does not appear in the partition function. In the wave function, there would be also cocycles and additional part linear in b_1 as presented in the previous section.

Now we consider the stationary configuration of the Euclidean path integral. We use the monopole solution $\mathcal{Z}, \bar{\mathcal{Z}}, \mathcal{A} = \tilde{A}$ as the field configuration and calculate the action. This illuminates the finer points of the wave function and cocycles. In this case, the Euclidean Chern-Simons action also vanishes. The wave function at infinity is almost abelian and the cocycle will be approximated by the previous appendix,

$$S_{\rm E} + 2\pi i \alpha_0(\phi) + ikm(-\partial_1^{-1} b_1 + \theta) \,. \tag{B.9}$$

The cocycle contribution vanishes since it is linear in $b = A - \tilde{A}$ and b vanishes for the present field configuration. The only possible contribution should arise from the wave function.

For the solution $\mathcal{Z} = (z_1, z_2)$ and $\bar{\mathcal{Z}} = (\bar{z}_1, \bar{z}_2)$, the asymptotic value of the solution from equation (3.13) becomes

$$\begin{aligned} \langle z_1 \rangle &= u_1 e^{i\theta_1} = \sqrt{u_1 u_2}, \ \langle z_2 \rangle = u_2 e^{i\theta_2} = \sqrt{u_1 u_2} \,, \\ \langle \bar{z}_1 \rangle &= u_1 e^{-i\theta_1} = \sqrt{\frac{u_1}{u_2}}, \ \langle \bar{z}_2 \rangle = u_2 e^{-i\theta_2} = \sqrt{\frac{u_2}{u_1}} \,. \end{aligned} \tag{B.10}$$

Thus the asymptotic value of the phase becomes imaginary

$$e^{-i\theta_1} = e^{i\theta_2} = e^{\Lambda_*} = \sqrt{\frac{u_1}{u_2}}\,. \tag{B.11}$$

For the BPS solutions $\mathcal{Z}, \bar{\mathcal{Z}}, \mathcal{A} = \bar{\mathcal{A}}$ of m monopoles, the matter action, the Chern-Simons term and the cocycles all vanish except the phase term from the wave function which is imaginary, or

$$e^{-S_{\rm E}} = e^{ikm(\theta_1-\theta_2)} = \left(\frac{u_1}{u_2}\right)^{km}. \tag{B.12}$$

C. Monopole vertex operator in the ABJM model

As discussed in section 4 and also in [35], the vertex operators are widely used to describe the low-energy effective interactions induced by monopole-instanton solutions. For the ABJM model, the monopole instanton vertex operators carry both magnetic flux and electric charge and would be different from those in three-dimensional Maxwell theory. We discuss in this section the monopole vertex operators in more details with emphasis on their physical origin.

Let us start with the flux creation operator $\Omega(\mathbf{x})$ in three-dimensional Maxwell theory whose UV description is the Georgi-Glashow model. It is well-known that an operator $\Omega(\mathbf{x})$ creating flux $\mathcal{B}$ at a point $\mathbf{x}$ takes the form as

$$\Omega(\mathbf{x}) = \exp\left(i\frac{\mathcal{B}}{4\pi}\sigma(\mathbf{x})\right), \tag{C.1}$$

where σ denotes the dual photon

$$F_{\mu\nu} = \frac{1}{4\pi}\epsilon_{\mu\nu\rho}\partial^\rho\sigma\,, \qquad [\sigma(\mathbf{x}), \partial_0\sigma(\mathbf{y})]_{\rm ET} = 16\pi^2 i\delta(\mathbf{x}-\mathbf{y})\,. \tag{C.2}$$

Here σ is normalized to have period 2π. One can show that $\Omega(\mathbf{x})$ creates a flux $\mathcal{B}$ at $\mathbf{x}$ using the relation $\partial_0\sigma = 4\pi F_{12}$ together with canonical equal-time commutation relation,

$$[F_{12}(\mathbf{x}), \Omega(\mathbf{y})] = \frac{1}{4\pi}[\partial_0\sigma(\mathbf{x}), \Omega(\mathbf{y})] = \mathcal{B}\delta(\mathbf{x}-\mathbf{y})\Omega(\mathbf{y})\,. \tag{C.3}$$

For the monopole-instanton that creates the flux $4\pi m$, the vertex operator becomes

$$\Omega_{\rm monopole}(\mathbf{x}) = \exp\left(im\sigma(\mathbf{x})\right)\,. \tag{C.4}$$

We now in turn consider the flux creation operator in the ABJM model whose low-energy dynamics can be effectively described as the BF-theory (2.12). It is not guaranteed that the flux creation operator in the BF-theory takes the same form as the previous one. We will show this is still the case. Let us restrict our attentions on a simple and illustrative BF-model

$$\mathcal{L} = -|D_\mu z|^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}b_\mu\partial_\nu c_\rho \tag{C.5}$$

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with $D_\mu z = \partial_\mu z - i b_\mu z$. The canonical commutation relation reads

$$[b_i(\mathbf{x}), c_j(\mathbf{y})]_{\mathrm{ET}} = +i\frac{4\pi}{k}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y})\,, \tag{C.6}$$

once we choose the temporal gauge $b_0 = c_0 = 0$. The Gauss laws become

$$\frac{k}{4\pi}F^{(+)}_{12} - \rho_B = 0\,, \qquad F^{(-)}_{12} = 0\,, \tag{C.7}$$

where ρ_B denote the gauge charge density and $F^{(+)} = dc$, $F^{(-)} = db$. They simply imply that we can identify the flux $F^{(+)}$ as the asymptotic unbroken U(1) field of the instanton which carries the electric charges. For the vertex operator of instanton, we therefore construct a certain operator $\Omega(\mathbf{x})$ that creates flux $F^{(+)}$ and charges.

In order to find out $\Omega(\mathbf{x})$ of our interest, we first introduce the Lagrangian multiplier

$$\mathcal{L} = -|D_\mu z|^2 + \frac{1}{8\pi}\epsilon^{\mu\nu\rho}\left(kb_\mu + \partial_\mu\sigma\right)F^{(+)}_{\nu\rho}\ . \tag{C.8}$$

The modified Lagrangian is invariant under the U(1) gauge symmetry

$$b_\mu \to b_\mu + \partial_\mu\lambda\,, \quad \sigma \to \sigma - k\lambda\,, \quad z \to e^{i\lambda}z\ . \tag{C.9}$$

Since, from (C.8), we can identify $\frac{1}{4\pi}F^{(+)}_{12}$ as the conjugate momentum of dual photon σ, the flux creation operator can be described as

$$\Omega^0(\mathbf{x}) = \exp\left(i\frac{\mathcal{B}}{4\pi}\sigma(\mathbf{x})\right)\ . \tag{C.10}$$

It however transforms under the gauge symmetry:

$$\Omega^0(\mathbf{x}) \ \to\ \exp\Big(-i\frac{k\mathcal{B}}{4\pi}\lambda(\mathbf{x})\Big)\Omega^0(\mathbf{x})\ . \tag{C.11}$$

We therefore conclude that, for gauge-invariance, the flux creation operator also needs the creation of $k\mathcal{B}/4\pi$ units of charges so as to satisfy the Gauss law (C.7). The gauge-invariant charge-flux creation operator $\Omega(\mathbf{x})$ thus takes the following form

$$\Omega(\mathbf{x}) = \Omega^0(\mathbf{x})\cdot\mathcal{Q}(\mathbf{x})\,, \tag{C.12}$$

where $\mathcal{Q}(\mathbf{x})$ carries the charges $\mathcal{B}k/4\pi$ so that its local gauge transformation is opposite to that of $\Omega^0(\mathbf{x})$. For the monopole-instanton that creates the flux $4\pi m$, the vertex operator becomes

$$\Omega_{\text{monopole}}(\mathbf{x}) = \exp\Big(im\sigma(\mathbf{x})\Big)\mathcal{Q}(\mathbf{x})\ , \tag{C.13}$$

where the operator $\mathcal{Q}(\mathbf{x})$ creates charge of mk.

These ideas can be applied to the ABJM model to explain the charge-flux creation operators (6.10) and the monopole vertex operators (6.11).

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